Application of the Information Bottleneck to LPAD Learning

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1 Application of the Information Bottleneck to LPAD Learning

In order to apply the Information Bottleneck (IB) to LPADs, the network $\mathcal{G}_{out}$ is the result of the translation of the LPAD for which we want to learn the parameters plus the addition of the Y variable. The set of hidden variables contains the vector of the choice variables $\mathbf{CH}$ plus those atoms that are unobserved in the data, let us call them $\mathbf{T}$. With $\mathbf{X}$ we indicate the set of atom variables that are observed in the data.

Suppose you want to learn the parameters of an LPAD using IB. Consider the LPAD L:
$r_1 = x1 : 0.4 \vee x2 : 0.3.$
$r_2 = x2 : 0.1 \vee x3 : 0.2.$
$r_3 = x4 : 0.6 \vee x5 : 0.4 \leftarrow x1.$
$r_4 = x5 : 0.4 \leftarrow x2, x3.$
$r_5 = x6 : 0.3 \vee x7 : 0.2 \leftarrow x2, x5.$
The Bayesian network equivalent to L is shown in Figure 1. In order to apply IB to LPADs, the

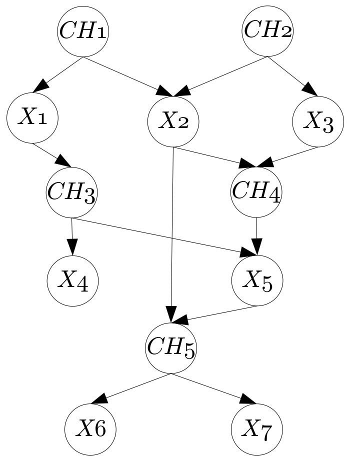

Fig. 1. Bayesian network

network $\mathcal{G}_{out}$ is the result of the translation of the LPAD for which we want to learn the parameters plus the addition of the Y variable.

For the $\mathcal{G}_{in}$ network, we consider a naive Bayes factorization:

$$Q(\mathbf{CH}, \mathbf{T}|Y) = \prod_i Q(CH_i|Y) \prod_j Q(T_j|Y).$$

In $\mathcal{G}_{out}$, the choice variables and the unobserved variables are the only parents of Y. In fact, given the choice variables and the unobserved variables, the $\mathbf{X}$ variables are uniquely determined,

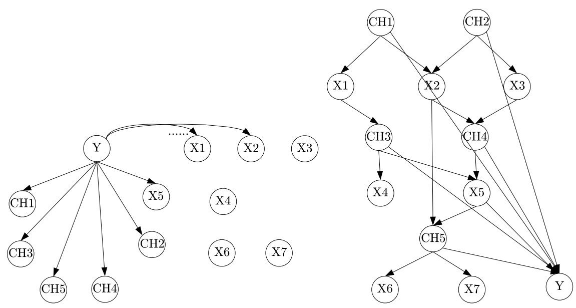

Fig. 2. $G_{in} = Q$ (left) and $G_{out} = P$ (right)

and so is the instance identity (assuming there are no duplicate examples, but this can be modeled by assigning them a different prior probability $Q(Y)$).

Consider the LPAD L. Moreover, suppose that x_5 is unseen in the data. The networks $\mathcal{G}_{in}$ and $\mathcal{G}_{out}$ for this LPAD are shown in Figure 2. According to IB, the chosen Q distribution must be such that unobserved variables are independent of observed ones given Y. This requirement is satisfied by $\mathcal{G}_{in}$ in Figure 2. As regards P, $\mathbf{CH}$ and $\mathbf{T}$ must be the only parents of Y. This requirement is also satisfied by $\mathcal{G}_{out}$: in fact, the observed variables are completely determined by knowing $\mathbf{CH}$ and $\mathbf{T}$ and so it the instance identity.

Note that for the network to be well defined the LPAD must be acyclic.

A ground normal logic program is *acyclic* [1] if the ground atoms can be assigned an integer level so that the level of the atom in the head of each rule is higher than the level of each atom in the body.

A disjunctive logic program is *acyclic* if the ground atoms can be assigned an integer level so that the level of each the atom in the head of each rule is the same and it is higher than the level of each atom in the body.

Let us compute the objective function $\mathcal{L}_{EM}$ for such a case

$$\mathcal{L}_{EM} = \mathbf{I}_Q(\mathbf{CH}, \mathbf{T}; Y) - \gamma(\mathbb{E}_Q[\log P(\mathbf{X}, \mathbf{T}, \mathbf{CH})] - \mathbb{E}_Q[\log Q(\mathbf{CH}, \mathbf{T})]) \tag{1}$$

where

$$\begin{aligned}\mathbf{I}_Q(\mathbf{CH}, \mathbf{T}; Y) = & \frac{H(\mathbf{CH}, \mathbf{T}) - H(\mathbf{CH}, \mathbf{T}|Y)}{\log_2 e} = \\ & -\mathbb{E}_Q[\log Q(\mathbf{CH}, \mathbf{T})] + \mathbb{E}_Q[\log Q(\mathbf{CH}, \mathbf{T}|Y)] = \\ & -\mathbb{E}_Q[\log Q(\mathbf{CH}, \mathbf{T})] + \sum_i \mathbb{E}_Q[\log Q(CH_i|Y)] + \sum_i + \mathbb{E}_Q[\log Q(T_i|Y)]\end{aligned}$$

and

$$\mathbb{E}_Q[\log Q(\mathbf{CH}, \mathbf{T})] \approx \sum_i \mathbb{E}_Q[\log Q(CH_i)] + \sum_i \mathbb{E}_Q[\log Q(T_i)]$$

following [2]. Thus

$$\begin{aligned}\mathcal{L}_{EM} = & \mathbb{E}_Q[\log Q(\mathbf{CH}, \mathbf{T}|Y)] - \gamma \mathbb{E}_Q[\log P(\mathbf{X}, \mathbf{T}, \mathbf{CH})] + (\gamma - 1)\mathbb{E}_Q[\log Q(\mathbf{CH}, \mathbf{T})]) = \\ & \sum_i \mathbb{E}_Q[\log Q(CH_i|Y)] + \sum_i \mathbb{E}_Q[\log Q(T_i|Y)] - \gamma \mathbb{E}_Q[\log P(\mathbf{X}, \mathbf{T}, \mathbf{CH})] + \\ & (\gamma - 1) \sum_i \mathbb{E}_Q[\log Q(ch_i)] + (\gamma - 1) \sum_i \mathbb{E}_Q[\log Q(t_i)]\end{aligned}$$

2 Stationary Points of $\mathcal{L}_{EM}$

In this section we describe how to find the stationary poins of $\mathcal{L}_{EM}$.

Proposition 1 (Stationary points of $\mathcal{L}_{EM}$). *Let $\mathcal{L}_{EM}$ be the function (1). $Q(ch_i|y)$ and $Q(t_i|y)$ are stationary points of $\mathcal{L}_{EM}$ with respect to a fixed choice of P if and only if for all values ch_i, t_i and y:*

$$Q(ch_i|y) = \frac{1}{Z_{\mathbf{CH}}(i,y,\gamma)} Q(ch_i)^{1-\gamma} e^{\gamma \mathbb{EP}(ch_i,y)},$$
$$Q(t_i|y) = \frac{1}{Z_{\mathbf{T}}(i,y,\gamma)} Q(t_i)^{1-\gamma} e^{\gamma \mathbb{EP}(t_i,y)}$$

where

$$Z_{\mathbf{CH}}(i,y,\gamma) = \sum_{ch'_i} Q(ch'_i)^{1-\gamma} e^{\gamma \mathbb{EP}(ch'_i,y)},$$
$$Z_{\mathbf{T}}(i,y,\gamma) = \sum_{t'_i} Q(t'_i)^{1-\gamma} e^{\gamma \mathbb{EP}(t'_i,y)}$$

are normalizing constants, and

$$\mathbb{EP}(ch_i,y) = \mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|ch_i,y)}[\log P(\mathbf{x}[y],\mathbf{T},\mathbf{CH})],$$

$$\mathbb{EP}(t_i,y) = E_{Q(\mathbf{CH},\mathbf{T}|t_i,y)}[\log P(\mathbf{x}[y],\mathbf{T},\mathbf{CH})].$$

Proof. We want to find the stationary points of $Q(ch_i|y)$ and $Q(t_i|y)$ under the constraint that

$$\sum_{ch_i} Q(ch_i|y) = 1$$

$$\sum_{t_i} Q(t_i|y) = 1$$

for any y.

Using Langrange multipliers, we want to optimize:

$$\mathcal{L} = \mathbb{E}_Q[\log Q(\mathbf{CH},\mathbf{T}|Y)] - \gamma(\mathbb{E}_Q[\log P(\mathbf{X},\mathbf{T},\mathbf{CH})] + \mathbb{E}_Q[\log Q(\mathbf{CH},\mathbf{T})])(\gamma-1) +$$
$$+ \sum_i \sum_y \lambda^{\mathbf{CH}}_{i,y} \left(\sum_{ch'_i} Q(ch'_i|y) - 1 \right) + \sum_i \sum_y \lambda^{\mathbf{T}}_{i,y} \left(\sum_{t'_i} Q(t'_i|y) - 1 \right).$$

Computing the derivative of $\mathcal{L}$ with respect to $Q(ch_{i0}|y_0)$ and $Q(t_{i0}|y_0)$ we get

$$\frac{\partial \mathbb{E}_Q[\log Q(CH_k|Y)]}{\partial Q(ch_{i0}|y_0)} = \frac{\partial \sum_{ch_k,y} Q(ch_k,y)\log Q(ch_k|y)}{\partial Q(ch_{i0}|y_0)} =$$
$$\left(Q(y_0)\log Q(ch_k|y_0) + \frac{Q(ch_k,y_0)}{Q(ch_k|y_0)} \right) 1\{i0=k, ch_{i0}=ch_k\} =$$
$$Q(y_0)(\log Q(ch_k|y_0)+1)1\{i0=k, ch_{i0}=ch_k\}$$

$$\frac{\partial \mathbb{E}_Q[\log Q(T_k|Y)]}{\partial Q(t_{i0}|y_0)} = \frac{\partial \sum_{t_k,y} Q(t_k,y)\log Q(t_k|y)}{\partial Q(t_{i0}|y_0)} =$$

$$\left(Q(y_0)\log Q(t_k|y_0) + \frac{Q(t_k,y_0)}{Q(t_k|y_0)}\right) 1\{i0 = k, t_{i0} = t_k\} =$$

$$Q(y_0)(\log Q(t_k|y_0) + 1) 1\{i0 = k, t_{i0} = t_k\}$$

$$\frac{\partial \mathbb{E}_Q[\log P(\mathbf{X},\mathbf{T},\mathbf{CH})]}{\partial Q(ch_{i0}|y_0)} = \frac{\partial \sum_{\mathbf{ch},\mathbf{t},y,\mathbf{x}} Q(\mathbf{ch},\mathbf{t},\mathbf{x},y)\log P(\mathbf{x},\mathbf{t},\mathbf{ch})}{\partial Q(ch_{i0}|y_0)} =$$

$$\sum_{\mathbf{ch},\mathbf{t},y,\mathbf{x}} \frac{\partial Q(\mathbf{ch},\mathbf{t},\mathbf{x},y)}{\partial Q(ch_{i0}|y_0)} \log P(\mathbf{x},\mathbf{t},\mathbf{ch}) =$$

$$\sum_{\mathbf{ch},\mathbf{t},y,\mathbf{x}} \frac{\partial Q(\mathbf{ch},\mathbf{t},\mathbf{x}|ch_i,y)Q(ch_i|y)Q(y)}{\partial Q(ch_{i0}|y_0)} \log P(\mathbf{x},\mathbf{t},\mathbf{ch}) =$$

$$\sum_{\mathbf{ch}\notin\{ch_i\},\mathbf{t},\mathbf{x}} Q(\mathbf{ch},\mathbf{t},\mathbf{x}|ch_{i0},y_0)Q(y_0)\log P(\mathbf{x},\mathbf{t},\mathbf{ch}) =$$

$$Q(y_0)\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|ch_{i0},y_0)}[\log P(\mathbf{x}[y_0],\mathbf{T},\mathbf{CH})]$$

$$\frac{\partial \mathbb{E}_Q[\log P(\mathbf{X},\mathbf{T},\mathbf{CH})]}{\partial Q(t_{i0}|y_0)} = Q(y_0)E_{Q(\mathbf{CH},\mathbf{T}|t_{i0},y_0)}[\log P(\mathbf{x}[y_0],\mathbf{T},\mathbf{CH})]$$

$$\frac{\partial \mathbb{E}_Q[\log Q(\mathbf{CH},\mathbf{T})]}{\partial Q(ch_{i0}|y_0)} =$$

$$\sum_k \sum_{ch_k} \frac{\partial Q(ch_k)\log Q(ch_k)}{\partial Q(ch_{i0}|y_0)} + \sum_k \sum_{t_k} \frac{\partial Q(t_k)\log Q(t_k)}{\partial Q(ch_{i0}|y_0)} =$$

$$\sum_k \sum_{ch_k} \frac{\partial Q(ch_k)}{\partial Q(ch_{i0}|y_0)} \log Q(ch_k) + \sum_{ch_k} Q(ch_k)\frac{\partial \log Q(ch_k)}{\partial Q(ch_{i0}|y_0)} +$$

$$\sum_k \sum_{t_k} \frac{\partial Q(t_k)}{\partial Q(ch_{i0}|y_0)} \log Q(t_k) + \sum_{t_k} Q(t_k)\frac{\partial \log Q(t_k)}{\partial Q(ch_{i0}|y_0)} =$$

$$\sum_k \sum_{ch_k} \frac{\partial \sum_y Q(ch_k|y)Q(y)}{\partial Q(ch_{i0}|y_0)} \log Q(ch_k) + \sum_{ch_k} Q(ch_k)\frac{\partial \log Q(ch_k)}{\partial Q(ch_{i0}|y_0)} +$$

$$\sum_k \sum_{t_k} \frac{\partial \sum_y Q(t_k|y)Q(y)}{\partial Q(ch_{i0}|y_0)} \log Q(t_k) + \sum_{t_k} Q(t_k)\frac{\partial \log Q(t_k)}{\partial Q(ch_{i0}|y_0)} =$$

$$\sum_k \sum_{ch_k} \frac{\partial \sum_y Q(ch_k|y)Q(y)}{\partial Q(ch_{i0}|y_0)} \log Q(ch_k) + \sum_{ch_k} Q(ch_k)\frac{1}{Q(ch_k)}\frac{\partial \sum_y Q(ch_k|y)Q(y)}{\partial Q(ch_{i0}|y_0)} +$$

$$\sum_k \sum_{t_k} \frac{\partial \sum_y Q(t_k|y)Q(y)}{\partial Q(ch_{i0}|y_0)} \log Q(t_k) + \sum_{t_k} Q(t_k)\frac{1}{Q(t_k)}\frac{\partial \sum_y Q(t_k|y)Q(y)}{\partial Q(ch_{i0}|y_0)} =$$

$$Q(y_0)\log Q(ch_{i0}) + Q(ch_{i0})\frac{Q(y_0)}{Q(ch_{i0})} = Q(y_0)(\log Q(ch_{i0}) + 1)$$

$$\frac{\partial \mathbb{E}_Q[\log Q(\mathbf{CH},\mathbf{T})]}{\partial Q(t_{i0}|y_0)} = Q(y_0)(\log Q(t_{i0}) + 1)$$

Therefore

$$\frac{\partial L}{\partial Q(ch_{i0}|y_0)} = Q(y_0)(\log Q(ch_{i0}|y_0)+1) - \gamma Q(y_0)\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|ch_{i0},y_0)}[\log P(\mathbf{x}[y_0],\mathbf{T},\mathbf{CH})] +$$
$$(\gamma-1)Q(y_0)(\log Q(ch_{i0})+1) + \lambda^{\mathbf{CH}}_{i,y_0} =$$
$$Q(y_0)(\log Q(ch_{i0}|y_0)+1-\gamma\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|ch_{i0},y_0)}[\log P(\mathbf{x}[y_0],\mathbf{T},\mathbf{CH})] +$$
$$(\gamma-1)(\log Q(ch_{i0})+1)) + \lambda^{\mathbf{CH}}_{i,y_0} =$$
$$Q(y_0)(\log Q(ch_{i0}|y_0)+1-\gamma\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|ch_{i0},y_0)}[\log P(\mathbf{x}[y_0],\mathbf{T},\mathbf{CH})] +$$
$$\gamma(\log Q(ch_{i0})+1) - \log Q(ch_{i0}) - 1) + \lambda^{\mathbf{CH}}_{i,y_0} =$$
$$Q(y_0)(\log Q(ch_{i0}|y_0) - \gamma\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|ch_{i0},y_0)}[\log P(\mathbf{x}[y_0],\mathbf{T},\mathbf{CH})] +$$
$$\gamma\log Q(ch_{i0}) + \gamma - \log Q(ch_{i0})) + \lambda^{\mathbf{CH}}_{i,y_0}$$

$$\frac{\partial L}{\partial Q(t_{i0}|y_0)} = Q(y_0)(\log Q(t_{i0}|y_0) - \gamma\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|t_{i0},y_0)}[\log P(\mathbf{x}[y_0],\mathbf{T},\mathbf{CH})] +$$
$$\gamma\log Q(t_{i0}) + \gamma - \log Q(t_{i0})) + \lambda^{\mathbf{T}}_{i,y_0}$$

Dividing by $Q(y_0)$ and equating to 0

$$Q(ch_{i0}|y_0) = e^{-\gamma-\lambda^{\mathbf{CH}}_{i,y_0}/Q(y_0)}Q(ch_{i0})^{1-\gamma}e^{\gamma\mathbb{EP}(ch_{i0},y_0)}$$
$$Q(t_{i0}|y_0) = e^{-\gamma-\lambda^{\mathbf{T}}_{i,y_0}/Q(y_0)}Q(t_{i0})^{1-\gamma}e^{\gamma\mathbb{EP}(t_{i0},y_0)}$$

where

$$\mathbb{EP}(ch_i,y) = \mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|ch_i,y)}[\log P(\mathbf{x}[y],\mathbf{T},\mathbf{CH})]$$
$$\mathbb{EP}(t_i,y) = E_{Q(\mathbf{CH},\mathbf{T}|t_i,y)}[\log P(\mathbf{x}[y],\mathbf{T},\mathbf{CH})]$$

Normalizing we get

$$Q(ch_i|y) = \frac{1}{Z_{\mathbf{CH}}(i,y,\gamma)}Q(ch_i)^{1-\gamma}e^{\gamma\mathbb{EP}(ch_i,y)}$$
$$Q(t_i|y) = \frac{1}{Z_{\mathbf{T}}(i,y,\gamma)}Q(t_i)^{1-\gamma}e^{\gamma\mathbb{EP}(t_i,y)}$$

where

$$Z_{\mathbf{CH}}(i,y,\gamma) = \sum_{ch'_i} Q(ch'_i)^{1-\gamma}e^{\gamma\mathbb{EP}(ch'_i,y)}$$
$$Z_{\mathbf{T}}(i,y,\gamma) = \sum_{t'_i} Q(t'_i)^{1-\gamma}e^{\gamma\mathbb{EP}(t'_i,y)}$$

□

3 Derivatives of $G_{ch_i,y}(Q,\gamma)$ and $G_{t_j,y}(Q,\gamma)$

Let us now compute the continuation direction. The G functions are

$$G_{ch_k,y}(Q,\gamma) = -\log Q(ch_k|y) + (1-\gamma)\log Q(ch_k) + \gamma\mathbb{EP}(ch_k,y) - \log Z_{\mathbf{CH}}(k,y,\gamma)$$

$$G_{t_k,y}(Q,\gamma) = -\log Q(t_k|y) + (1-\gamma)\log Q(t_k) + \gamma\mathbb{EP}(t_k,y) - \log Z_{\mathbf{T}}(k,y,\gamma)$$

We apply a computation similar to the one in appendix B in [3].

Let us first express the parameters of P: $\theta_{x_j|pa_{x_j}} = 1$ if $x_j \in val(\mathbf{pa}_{x_j})$ and 0 otherwise, where $val(\mathbf{pa}_{x_j})$ is the set of values assumed by $\mathbf{pa}_{x_j}$. For a non ground rule r, let $\theta_{Hd_r=hd_r|body_r}$ or simply $\theta_{hd_r|body_r}$ be the probability that the head hd_r is selected given that the body has truth value $body_r$. Let $i(r)$ be the set of instances of r and, given a ground rule k, let $r(k)$ be the non ground rule of which k is an instance. Given the body $\mathbf{pa}_{ch_k}$ of instantiated rule k, let $bt(\mathbf{pa}_{ch_k})$ be 1 if the observed variables in $\mathbf{pa}_{ch_k}$ do not make the body false and 0 otherwise. Let $tb(\mathbf{pa}_{ch_k})$ be a set of values for the unobserved variables that are parents of ch_k. The values are those that do not make the body false.

The values of $\theta_{hd_r|body_r}$ are

$$\theta_{Hd_r=null|false} = 1.0$$

$$\theta_{Hd_r=x|true} = 0.0$$

$$\theta_{hd_r|true} = \frac{\sum_{s\in i(r)} \sum_y Q(y)Q(Ch_s = hd_r|y)bt(\mathbf{pa}_{ch_s}[y]) \prod_{t_i\in tb(\mathbf{pa}_{ch_s})} Q(t_i|y) + \alpha(r, hd_r, true)}{\sum_{s\in i(r)} \sum_y Q(y)bt(\mathbf{pa}_{ch_s}[y]) \prod_{t_i\in tb(\mathbf{pa}_{ch_s})} Q(t_i|y) + \alpha(r, true)} =$$

$$\frac{\sum_{s,ch_s\in i(r)} \mathcal{N}(s, hd_r) + \alpha(r, hd_r, true)}{\mathcal{N}(r)} =$$

$$\frac{\mathcal{N}(r, hd_r)}{\mathcal{N}(r)}$$

Let

$$\mathcal{N}(s, hd_r) = \sum_y Q(y)Q(Ch_s = hd_r|y)bt(\mathbf{pa}_{ch_s}[y]) \prod_{t_i\in tb(\mathbf{pa}_{ch_s})} Q(t_i|y)$$

The derivatives of the parameters are:

$$\frac{\partial \mathcal{N}(s, hd_r)}{\partial Q(ch_{i0}|y_0)} = Q(y_0)bt(\mathbf{pa}_{ch_s}[y_0]) \prod_{t_j\in tb(\mathbf{pa}_{ch_s})} Q(t_j|y_0)1\{ch_{i0} = hd_r\}$$

$$\frac{\partial \mathcal{N}(r, hd_r)}{\partial Q(ch_{i0}|y_0)} = Q(y_0) \sum_{s\in i(r)} bt(\mathbf{pa}_{ch_s}[y_0]) \prod_{t_j\in tb(\mathbf{pa}_{ch_s})} Q(t_j|y_0)1\{ch_{i0} = hd_r\}$$

$$\frac{\partial \mathcal{N}(r)}{\partial Q(ch_{i0}|y_0)} = 0$$

$$\frac{\partial \mathcal{N}(s, hd_r)}{\partial Q(t_{i0}|y_0)} = Q(y_0)Q(Ch_s = hd_r|y_0)bt(\mathbf{pa}_{ch_s}[y_0]) \prod_{t_i\in tb(\mathbf{pa}_{ch_s})\setminus\{t_{i0}\}} Q(t_i|y_0)1\{t_{i0} \in tb(\mathbf{pa}_{ch_s})\}$$

$$\frac{\partial \mathcal{N}(r, hd_r)}{\partial Q(t_{i0}|y_0)} = Q(y_0) \sum_{s\in i(r)} Q(Ch_s = hd_r|y_0)bt(\mathbf{pa}_{ch_s}[y_0]) \prod_{t_j\in tb(\mathbf{pa}_{ch_s})\setminus\{t_{i0}\}} Q(t_j|y_0)1\{t_{i0} \in tb(pa_{ch_s})\}$$

$$\frac{\partial \mathcal{N}(r)}{\partial Q(t_{i0}|y_0)} = Q(y_0) \sum_{s\in i(r)} bt(\mathbf{pa}_{ch_s}[y_0]) \prod_{t_j\in tb(\mathbf{pa}_{ch_s})\setminus\{t_{i0}\}} Q(t_j|y_0)1\{t_{i0} \in tb(\mathbf{pa}_{ch_s})\}$$

$$\frac{\partial \theta_{x_j|\mathbf{pa}_{x_j}}}{\partial Q(ch_{i0}|y_0)} = 0$$

$$\frac{\partial \theta_{t_k|\mathbf{pa}_{t_k}}}{\partial Q(ch_{i0}|y_0)} = 0$$

$$\frac{\partial\theta_{x_j|\mathbf{pa}_{x_j}}}{\partial Q(t_{i0}|y_0)} = 0$$

$$\frac{\partial\theta_{t_k|\mathbf{pa}_{t_k}}}{\partial Q(t_{i0}|y_0)} = 0$$

$$\frac{\partial\theta_{hd_r|false}}{\partial Q(ch_{i0}|y_0)} = 0$$

$$\frac{\partial\theta_{hd_r|true}}{\partial Q(ch_{i0}|y_0)} = \frac{\frac{\partial\mathcal{N}(r,hd_r)}{\partial Q(ch_{i0}|y_0)}\mathcal{N}(r) + \mathcal{N}(r,hd_r)\frac{\partial\mathcal{N}(r)}{\partial Q(ch_{i0}|y_0)}}{\mathcal{N}(r)^2} = Q(y_0)\frac{\sum_{s\in i(r)} bt(\mathbf{pa}_{ch_s}[y_0])\prod_{t_j\in tb(\mathbf{pa}_{ch_s})} Q(t_j|y_0)1\{ch_{i0} = hd_r\}}{\mathcal{N}(r)}$$

$$\frac{\partial\theta_{hd_r|true}}{\partial Q(t_{i0}|y_0)} = \frac{\frac{\partial\mathcal{N}(r,hd_r)}{\partial Q(t_{i0}|y_0)}\mathcal{N}(r) + \mathcal{N}(r,hd_r)\frac{\partial\mathcal{N}(r)}{\partial Q(t_{i0}|y_0)}}{\mathcal{N}(r)^2} = \frac{Q(y_0)\sum_{s\in i(r)} Q(Ch_s = hd_r|y_0)bt(\mathbf{pa}_{ch_s}[y_0])\prod_{t_j\in tb(\mathbf{pa}_{ch_s})\setminus\{t_{i0}\}} Q(t_j|y_0)1\{t_{i0}\in tb(\mathbf{pa}_{ch_s})\}}{\mathcal{N}(r)} + \frac{Q(y_0)\sum_{s\in i(r)} bt(\mathbf{pa}_{ch_s}[y_0])\prod_{t_j\in tb(\mathbf{pa}_{ch_s})\setminus\{t_{i0}\}} Q(t_j|y_0)1\{t_{i0}\in tb(\mathbf{pa}_{ch_s})\}\mathcal{N}(r,hd_r)}{\mathcal{N}(r)^2}$$

Let us now compute $\frac{\partial \log P(\mathbf{x}[y],\mathbf{t},\mathbf{ch})}{\partial Q(ch_{i0}|y_0)}$. We can express $\log P(\mathbf{x}[y],\mathbf{t},\mathbf{ch})$ with respect to ch_i as

$$\log P(\mathbf{x}[y],\mathbf{t},\mathbf{ch}) = \sum_{k\in t(i,y)} \log\theta_{Hd_{r(i)}=ch_k|true} + \sum_{k\notin t(i,y)} \log\theta_{Hd_{r(i)}=ch_k|\mathbf{pa}_{ch_k}}[y] + \sum_j \log\theta_{x_j|\mathbf{pa}_{x_j}}[y] + \sum_k \log\theta_{t_k|\mathbf{pa}_{t_k}}$$

where $t(i,y)$ is the set of indexes of choice variables ch_k that are instances of rule $r(i)$ and such that the instantiated rule k has the body true with respect to $\mathbf{x}[y],\mathbf{t}$. Since the summations $\sum_{k\notin t(i,y)}$ and $\sum_j$ have 0 derivative, it is possible to prove that

Proposition 2 (Derivatives of $\log P$ with respect to $Q(ch_{i0}|y_0)$).

$$\frac{\partial \log P(\mathbf{x}[y_0],\mathbf{t},\mathbf{ch})}{\partial Q(ch_{i0}|y_0)} = Q(y_0)\mathcal{D}(y_0, ch_{t(i,y_0)}, ch_{i0}, tb(\mathbf{pa}_{ch_{t(i,y_0)}}))$$

where $ch_{t(i,y)}$ is the set of choice variables ch_k with $k\in t(i,y)$, and

$$\mathcal{D}(y_0, ch_{t(i,y_0)}, ch_{i0}) = \frac{1}{\mathcal{N}(r(i))}\left(\sum_{k,k\in t(i,y_0)} \frac{1}{\theta_{Hd_{r(i)}=ch_k|true}}\right) N(i, ch_{t(i,y_0)}, ch_{i0}, tb(\mathbf{pa}_{ch_{t(i,y_0)}}))$$

with $N(i, ch_{t(i,y_0)}, ch_{i0}, tb(\mathbf{pa}_{ch_{t(i,y_0)}})) = \sum_{s\in t(i,y_0), ch_{i0}=ch_s}\prod_{t_j\in tb(\mathbf{pa}_{ch_s})} Q(t_j|y_0)$.

Proof.

$$\frac{\partial \log P(\mathbf{x}[y_0],\mathbf{t},\mathbf{ch})}{\partial Q(ch_{i0}|y_0)} =$$

$$\sum_{k\in t(i,y_0)} \frac{1}{\theta_{Hd_{r(i)}=ch_k|true}} \frac{\partial \theta_{Hd_{r(i)}=ch_k|true}}{\partial Q(ch_{i0}|y_0)} =$$

$$Q(y_0)\sum_{k\in t(i,y_0)} \frac{1}{\theta_{Hd_{r(i)}=ch_k|true}} \sum_{s\in t(i,y_0)} \frac{\prod_{t_j\in tb(\mathbf{pa}_{ch_s})} Q(t_j|y_0)1\{ch_{i0}=ch_s\}}{\mathcal{N}(r(i))} =$$

$$Q(y_0)\frac{1}{\mathcal{N}(r(i))}\sum_{k\in t(i,y_0)} \frac{1}{\theta_{Hd_{r(i)}=ch_k|true}} \sum_{s\in t(i,y_0)} \prod_{t_j\in tb(\mathbf{pa}_{ch_s})} Q(t_j|y_0)1\{ch_{i0}=ch_s\} =$$

$$Q(y_0)\frac{1}{\mathcal{N}(r(i))}\left(\sum_{k\in t(i,y_0)} \frac{1}{\theta_{Hd_{r(i)}=ch_k|true}}\right)\left(\sum_{s\in t(i,y_0)} \prod_{t_j\in tb(\mathbf{pa}_{ch_s})} Q(t_j|y_0)1\{ch_{i0}=ch_s\}\right) =$$

$$Q(y_0)\frac{1}{\mathcal{N}(r(i))}\left(\sum_{k\in t(i,y_0)} \frac{1}{\theta_{Hd_{r(i)}=ch_k|true}}\right)\left(\sum_{s\in t(i,y_0),ch_{i0}=ch_s} \prod_{t_j\in tb(\mathbf{pa}_{ch_s})} Q(t_j|y_0)\right) =$$

$$Q(y_0)\frac{1}{\mathcal{N}(r(i))}T(i,ch_{t(i,y_0)})N(i,ch_{t(i,y_0)},ch_{i0},tb(\mathbf{pa}_{ch_{t(i,y_0)}})) =$$

$$Q(y_0)\mathcal{D}(y_0,ch_{t(i,y_0)},ch_{i0},tb(\mathbf{pa}_{ch_{t(i,y_0)}}))$$

where $ch_{t(i,y)}$ is the set of choice variables ch_k with $k \in t(i,y)$. □

Let us now compute $\frac{\partial \log P(\mathbf{x}[y],\mathbf{t},\mathbf{ch})}{\partial Q(t_{i0}|y_0)}$. We can express $\log P(\mathbf{x}[y],\mathbf{t},\mathbf{ch})$ with respect to t_i as

$$\log P(\mathbf{x}[y],\mathbf{t},\mathbf{ch}) = \sum_{k\in b(t_i,y,\mathbf{x}[y],\mathbf{t})} \log\theta_{Hd_{r(k)}=ch_k|true} + \sum_{k\notin b(t_i,y,\mathbf{x}[y],\mathbf{t})} \log\theta_{Hd_{r(i)}=ch_k|\mathbf{pa}_{ch_k}}[y]+$$

$$\sum_j \log\theta_{x_j|\mathbf{pa}_{x_j}}[y] + + \sum_k \log\theta_{t_k|\mathbf{pa}_{t_k}}$$

Let $b(t_i,y,\mathbf{x}[y],\mathbf{t})$ be the set of indexes of instantiations of rules for which t_i appears in the body with a matching truth value and such that the body is true with respect to $\mathbf{x}[y],\mathbf{t}$. The summations $\sum_{k\notin b(t_i,y,\mathbf{x}[y],\mathbf{t})}$, $\sum_j$ and $\sum_k$ have 0 derivative with respect to $Q(t_{i0}|y_0)$.

Proposition 3 (Derivatives of $\log P$ with respect to $Q(t_{i0}|y_0)$).

$$\frac{\partial \log P(\mathbf{x}[y_0],\mathbf{t},\mathbf{ch})}{\partial Q(t_{i0}|y_0)} = Q(y_0)\mathcal{F}(y_0,ech(t_{i0}),et(t_{i0}))$$

where

$$\mathcal{F}(y_0,ech(t_{i0}),et(t_{i0})) =$$

$$Q(y_0)\sum_{k\in b(t_{i0},y_0,\mathbf{x}[y_0],\mathbf{t})}\sum_{s\in i(r(k))} Q(Ch'_s=ch_k|y_0)bt(\mathbf{pa}_{ch_s}[y_0])$$

$$\frac{\prod_{t'_j\in tb(\mathbf{pa}_{ch_s})\setminus t_{i0}} Q(t'_j|y_0)1\{t_{i0}\in tb(\mathbf{pa}_{ch_s})\}}{\mathcal{N}(r(k))}\left(\frac{1}{\theta_{Hd_{r(k)}=ch_k|true}}+1\right)$$

and $ech(t_{i0})$ is the set of ch_s where $s \in b(t_{i0},y_0,\mathbf{x}[y_0],\mathbf{t})$ and $et(t_{i0})$ is the set of t_i such that t_i appears in the body of a rule s with $s \in b(t_{i0},y_0,\mathbf{x}[y_0],\mathbf{t})$ with a matching truth value.

Proof.

$$\frac{\partial \log P(\mathbf{x}[y_0], \mathbf{t}, \mathbf{ch})}{\partial Q(t_{i0}|y_0)} =$$

$$\sum_{k \in b(t_{i0}, y_0, \mathbf{x}[y], \mathbf{t})} \frac{1}{\theta_{Hd_{r(k)}=ch_k|true}} \frac{\partial \theta_{Hd_{r(k)}=ch_k|true}}{\partial Q(t_{i0}|y_0)} =$$

$$Q(y_0) \sum_{k \in b(t_{i0}, y_0, \mathbf{x}[y_0], \mathbf{t})} \frac{1}{\theta_{Hd_{r(k)}=ch_k|true}} \cdot$$

$$\Bigg(\frac{Q(y_0) \sum_{s \in i(r(k))} Q(Ch'_s = ch_k|y_0) bt(\mathbf{pa}_{ch_s}[y_0]) \prod_{t'_j \in tb(\mathbf{pa}_{ch_s}) \setminus \{t_{i0}\}} Q(t'_j|y_0) 1\{t_{i0} \in tb(\mathbf{pa}_{ch_s})\}}{\mathcal{N}(r(k))} +$$

$$\frac{Q(y_0) \sum_{s \in i(r(k))} bt(\mathbf{pa}_{ch_s}[y_0]) \prod_{t'_j \in tb(\mathbf{pa}_{ch_s}) \setminus \{t_{i0}\}} Q(t'_j|y_0) 1\{t_{i0} \in tb(\mathbf{pa}_{ch_s})\} \mathcal{N}(r(k), ch_k)}{\mathcal{N}(r(k))^2} \Bigg) =$$

$$Q(y_0) \sum_{k \in b(t_{i0}, y_0, \mathbf{x}[y_0], \mathbf{t})} \frac{1}{\theta_{Hd_{r(k)}=ch_k|true}} \sum_{s \in i(r(k))} Q(Ch'_s = ch_k|y_0) bt(\mathbf{pa}_{ch_s}[y_0]) \cdot$$

$$\Bigg(\frac{\prod_{t'_j \in tb(\mathbf{pa}_{ch_s}) \setminus \{t_{i0}\}} Q(t'_j|y_0) 1\{t_{i0} \in tb(\mathbf{pa}_{ch_s})\}}{\mathcal{N}(r(k))} +$$

$$\frac{\prod_{t'_j \in tb(\mathbf{pa}_{ch_s}) \setminus \{t_{i0}\}} Q(t'_j|y_0) 1\{t_{i0} \in tb(\mathbf{pa}_{ch_s})\} \mathcal{N}(r(k), ch_k)}{\mathcal{N}(r(k))^2} \Bigg) =$$

$$Q(y_0) \sum_{k \in b(t_{i0}, y_0, \mathbf{x}[y_0], \mathbf{t})} \frac{1}{\theta_{Hd_{r(k)}=ch_k|true}} \sum_{s \in i(r(k))} Q(Ch'_s = ch_k|y_0) bt(\mathbf{pa}_{ch_s}[y_0]) \cdot$$

$$\frac{\prod_{t'_j \in tb(\mathbf{pa}_{ch_s}) \setminus \{t_{i0}\}} Q(t_j|y_0) 1\{t_{i0} \in tb(\mathbf{pa}_{ch_s})\}}{\mathcal{N}(r(k))} \left(1 + \frac{\mathcal{N}(r(k), ch_k)}{\mathcal{N}(r(k))} \right) =$$

$$Q(y_0) \sum_{k \in b(t_{i0}, y_0, \mathbf{x}[y_0], \mathbf{t})} \frac{1}{\theta_{Hd_{r(k)}=ch_k|true}} \sum_{s \in i(r(k))} Q(Ch'_s = ch_k|y_0) bt(\mathbf{pa}_{ch_s}[y_0]) \cdot$$

$$\frac{\prod_{t'_j \in tb(\mathbf{pa}_{ch_s}) \setminus \{t_{i0}\}} Q(t_j|y_0) 1\{t_{i0} \in tb(\mathbf{pa}_{ch_s})\}}{\mathcal{N}(r(k))} \left(1 + \theta_{Hd_{r(k)}=ch_k|true} \right) =$$

$$Q(y_0) \sum_{k \in b(t_{i0}, y_0, \mathbf{x}[y_0], \mathbf{t})} \sum_{s \in i(r(k))} Q(Ch'_s = ch_k|y_0) bt(\mathbf{pa}_{ch_s}[y_0]) \cdot$$

$$\frac{\prod_{t'_j \in tb(\mathbf{pa}_{ch_s}) \setminus t_{i0}} Q(t'_j|y_0) 1\{t_{i0} \in tb(\mathbf{pa}_{ch_s})\}}{\mathcal{N}(r(k))} \left(\frac{1}{\theta_{Hd_{r(k)}=ch_k|true}} + 1 \right) =$$

$$Q(y_0) \mathcal{F}(y_0, ech(t_{i0}), et(t_{i0}))$$

□

Proposition 4 (Derivatives of $\mathbb{EP}(ch_i, y_0)$ with respect to $Q(ch_{i0}|y_0)$).

$$\frac{\partial \mathbb{EP}(ch_i, y_0)}{\partial Q(ch_{i0}|y_0)} =$$

$$Q(y_0) \sum_{ch_k \in ch_t(i, y_0) \setminus \{ch_i\}, t \in tb(\mathbf{pa}_{ch_{t(i,y_0)}})} \prod_{ch_k \in ch_t(i, y_0) \setminus \{ch_i\}} Q(ch_k|y_0)$$

$$\prod_{t_i \in tb(\mathbf{pa}_{ch_{t(i,y_0)}})} Q(t_i|y_0)\mathcal{D}(y_0, ch_{t(i,y_0)}, ch_{i0}, tb(\mathbf{pa}_{ch_{t(i,y_0)}}))$$

Proof.

$$\begin{aligned}
\frac{\partial \mathbb{EP}(ch_i, y_0)}{\partial Q(ch_{i0}|y_0)} &= \frac{\partial \mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|ch_i,y_0)}[\log P(\mathbf{x}[y_0], \mathbf{T}, \mathbf{CH})]}{\partial Q(ch_{i0}|y_0)} \\
&= \frac{\partial \sum_{\mathbf{ch},\mathbf{t}} Q(\mathbf{ch}, \mathbf{t}|ch_i, y_0) \log P(\mathbf{x}[y_0], \mathbf{t}, \mathbf{ch})}{\partial Q(ch_{i0}|y_0)} \\
&= \sum_{\mathbf{ch}} \frac{\partial Q(\mathbf{ch}, \mathbf{t}|ch_i, y_0)}{\partial Q(ch_{i0}|y_0)} \log P(\mathbf{x}[y_0], \mathbf{t}, \mathbf{ch}) + Q(\mathbf{ch}, \mathbf{t}|ch_i, y_0) \frac{\partial \log P(\mathbf{x}[y_0], \mathbf{t}, \mathbf{ch})}{\partial Q(ch_{i0}|y_0)}
\end{aligned}$$

The first term is 0 because $Q(\mathbf{ch}, \mathbf{t}|ch_i, y_0)$ is constant with respect to $Q(ch_{i0}|y_0)$. Thus

$$\begin{aligned}
\frac{\partial \mathbb{EP}(ch_i, y_0)}{\partial Q(ch_{i0}|y_0)} &= E_{Q(\mathbf{CH},\mathbf{T}|ch_i,y_0)}[\frac{\partial \log P(\mathbf{x}[y_0], \mathbf{t}, \mathbf{ch})}{\partial Q(ch_{i0}|y_0)}]\{ch_i = ch_{i0}\} = \\
&E_{Q(\mathbf{CH},\mathbf{T}|ch_i,y_0)}[Q(y_0)\mathcal{D}(y_0, \mathbf{ch}_{t(i,y_0)}, ch_{i0}, tb(\mathbf{pa}_{ch_{t(i,y_0)}}))]\{ch_i = ch_{i0}\} = \\
&Q(y_0)\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|ch_i,\mathbf{pa}_{ch_i 0})}[\mathcal{D}(y_0, \mathbf{ch}_{t(i,y_0)}, ch_{i0}, tb(\mathbf{pa}_{ch_{t(i,y_0)}}))]\{ch_i = ch_{i0}\}
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|ch_{i0},y_0)}[\mathcal{D}(y_0, ch_{t(i,y_0)}, ch_{i0}), tb(\mathbf{pa}_{ch_{t(i,y_0)}})] = \\
&\textstyle\sum_{ch_k \in ch_t(i,y_0)\setminus\{ch_i\}, t \in tb(\mathbf{pa}_{ch_{t(i,y_0)}})} Q(ch_{t(i,y_0)} t_{tb(\mathbf{pa}_{ch_{t(i,y_0)}})}|y_0)\mathcal{D}(y_0, ch_{t(i,y_0)}, ch_{i0}, tb(\mathbf{pa}_{ch_{t(i,y_0)}})) = \\
&\quad\textstyle\sum_{ch_k \in ch_t(i,y_0)\setminus\{ch_i\}, t \in tb(\mathbf{pa}_{ch_{t(i,y_0)}})} \prod_{ch_k \in ch_t(i,y_0)\setminus\{ch_i\}} Q(ch_k|y_0) \\
&\quad\textstyle\prod_{t_i \in tb(\mathbf{pa}_{ch_{t(i,y_0)}})} Q(t_i|y_0)\mathcal{D}(y_0, ch_{t(i,y_0)}, ch_{i0}, tb(\mathbf{pa}_{ch_{t(i,y_0)}}))
\end{aligned}$$

□

Proposition 5 (Derivatives of $\mathbb{EP}(t_i, y_0)$ with respect to $Q(t_{i0}|y_0)$).

$$\begin{aligned}
&\frac{\partial \mathbb{EP}(t_i, y_0)}{\partial Q(t_{i0}|y_0)} = \\
&\frac{Q(y_0)}{Q(t_{i0}|y_0)} \sum_{k, t_i \in body(k))} \prod_{t_j \in body(k)\setminus\{t_i\}} Q(t_j|y_0) \cdot \\
&\sum_{ch_k} Q(ch_k|y_0) \sum_{s \in i(r(k))} 1\{t_{j0} \in tb(\mathbf{pa}_{ch_s})\} U(s, ch_k, y_0)
\end{aligned}$$

where

$$U(s, ch_k, y_0) = Q(Ch'_s = ch_k|y_0) bt(\mathbf{pa}_{ch_s}[y_0]) \frac{\prod_{t'_j \in tb(\mathbf{pa}_{ch_s})} Q(t'_j|y_0)}{\mathcal{N}(r(k))} \left(\frac{1}{\theta_{Hd_{r(k)}=ch_k|true}} + 1 \right)$$

Proof.

$$\begin{aligned}
\frac{\partial \mathbb{EP}(t_i, y_0)}{\partial Q(t_{i0}|y_0)} &= \mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|t_i,y_0)}[\frac{\partial \log P(\mathbf{x}[y_0], \mathbf{t}, \mathbf{ch})}{\partial Q(t_{i0}|y_0)}]\{t_i = t_{i0}\} = \\
&\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|t_i,y_0)}[Q(y_0)\mathcal{F}(y_0, ech(t_{i0}), et(t_{i0}))]\{t_i = t_{i0}\} = \\
&Q(y_0)\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|t_i,y_0)}[\mathcal{F}(y_0, ech(t_{i0}), et(t_{i0}))]\{t_i = t_{i0}\}
\end{aligned}$$

$$
\begin{aligned}
&\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|t_{i0},y_0)}[\mathcal{F}(y_0, ech(t_{i0}), et(t_{i0}))] = \\
&\sum_{\mathbf{ch},\mathbf{t}\setminus\{t_{i0}\}} Q(\mathbf{ch},\mathbf{t}|y_0)\mathcal{F}(y_0, ech(t_{i0}), et(t_{i0})) = \\
&\sum_{\mathbf{ch},\mathbf{t}\setminus\{t_{i0}\}} Q(\mathbf{ch},\mathbf{t}|y_0)Q(y_0) \sum_{k\in b(t_{i0},y_0,\mathbf{x}[y_0],\mathbf{t})} \sum_{s\in i(r(k))} Q(Ch'_s = ch_k|y_0)bt(\mathbf{pa}_{ch_s}[y_0]) \cdot \\
&\frac{\prod_{t'_j\in tb(\mathbf{pa}_{ch_s})\setminus\{t_{i0}\}} Q(t'_j|y_0)1\{t_{j0}\in tb(\mathbf{pa}_{ch_s})\}}{\mathcal{N}(r(k))}\left(\frac{1}{\theta_{Hd_{r(k)}=ch_k|true}}+1\right) = \\
&Q(y_0)\sum_{k,t_i\in body(k))}\prod_{t_j\in body(k)\setminus\{t_i\}} Q(t_j|y_0)\sum_{ch} Q(ch|y_0)\sum_{s\in i(r(k))} Q(Ch'_s = ch_k|y_0)bt(\mathbf{pa}_{ch_s}[y_0]) \cdot \\
&\frac{\prod_{t'_j\in tb(\mathbf{pa}_{ch_s})\setminus\{t_{i0}\}} Q(t'_j|y_0)1\{t_{j0}\in tb(\mathbf{pa}_{ch_s})\}}{\mathcal{N}(r(k))}\left(\frac{1}{\theta_{Hd_{r(k)}=ch_k|true}}+1\right) = \\
&Q(y_0)\sum_{k,t_i\in body(k))}\prod_{t_j\in body(k)\setminus\{t_i\}} Q(t_j|y_0)\sum_{ch_k} Q(ch_k|y_0)\sum_{s\in i(r(k))} Q(Ch'_s = ch_k|y_0)bt(\mathbf{pa}_{ch_s}[y_0]) \cdot \\
&\frac{\prod_{t'_j\in tb(\mathbf{pa}_{ch_s})\setminus\{t_{i0}\}} Q(t'_j|y_0)1\{t_{j0}\in tb(\mathbf{pa}_{ch_s})\}}{\mathcal{N}(r(k))}\left(\frac{1}{\theta_{Hd_{r(k)}=ch_k|true}}+1\right) = \\
&\frac{Q(y_0)}{Q(t_{i0}|y_0)}\sum_{k,t_i\in body(k))}\prod_{t_j\in body(k)\setminus\{t_i\}} Q(t_j|y_0)\cdot \\
&\sum_{ch_k} Q(ch_k|y_0)\sum_{s\in i(r(k))} 1\{t_{j0}\in tb(\mathbf{pa}_{ch_s})\}U(s, ch_k, y_0)
\end{aligned}
$$

where

$$
U(s, ch_k, y_0) = Q(Ch'_s = ch_k|y_0)bt(\mathbf{pa}_{ch_s}[y_0])\frac{\prod_{t'_j\in tb(\mathbf{pa}_{ch_s})} Q(t'_j|y_0)}{\mathcal{N}(r(k))}\left(\frac{1}{\theta_{Hd_{r(k)}=ch_k|true}}+1\right)
$$

□

Theorem 1 (Derivatives of G with respect to $Q(ch_i|y)$ and $Q(t_i|y)$). *Overall, the derivative of G with respect to ch_i and t_i are:*

$$
\begin{aligned}
&\frac{\partial G_{ch_i,y}(Q,\gamma)}{\partial Q(ch_i|y)} = \\
&-\frac{1}{Q(ch_i|y)} + Q(y)(1-Q(ch_i|y))\left(\frac{1-\gamma}{Q(ch_i)} + \gamma\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|ch_i,y_0)}[\mathcal{D}(y_0, ch_{t(i,y),ch_{i0}}, tb(\mathbf{pa}_{ch_{t(i,y_0)}}))]\right)
\end{aligned}
$$

$$
\frac{\partial G_{t_i,y}(Q,\gamma)}{\partial Q(t_i|y)} = -\frac{1}{Q(t_i|y)} + Q(y)(1-Q(t_i|y))\left(\frac{1-\gamma}{Q(t_i)} + \gamma\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|t_i,y_0)}[\mathcal{F}(y_0, ech(t_{i0}), et(t_{i0}))]\right)
$$

Proof.

$$
\frac{\partial \log Q(ch_i)}{\partial Q(ch_{i0}|y_0)} = \frac{Q(y_0)}{Q(ch_i)}1\{ch_i = ch_{i0}\}
$$

$$
\frac{\partial \log Q(t_i)}{\partial Q(t_{i0}|y_0)} = \frac{Q(y_0)}{Q(t_i)}1\{t_i = t_{i0}\}
$$

In the following we need

$$\frac{\partial((1-\gamma)\log Q(ch_i)+\gamma\mathbb{EP}(ch_i,y_0))}{\partial Q(ch_{i0}|y_0)} =$$
$$Q(y_0)\left(\frac{1-\gamma}{Q(ch_i)}1\{ch_i=ch_{i0}\}+\gamma\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|ch_i,y_0)}[\mathcal{D}(y_0,ch_{t(i,y),ch_{i0}},tb(\mathbf{pa}_{ch_{t(i,y_0)}}))]\right)$$

$$\frac{\partial((1-\gamma)\log Q(t_i)+\gamma\mathbb{EP}(t_i,y_0))}{\partial Q(t_{i0}|y_0)} =$$
$$Q(y_0)\left(\frac{1-\gamma}{Q(t_i)}1\{t_i=t_{i0}\}+\gamma\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|t_i,y_0)}[\mathcal{F}(y_0,ech(t_{i0}),et(t_{i0}))]\right)$$

Let us write $\log Z_{\mathbf{CH}}(i,y_0,\gamma)$

$$\log Z_{\mathbf{CH}}(i,y_0,\gamma)=\log\sum_{ch'_i}e^{(1-\gamma)\log Q(ch'_i)+\gamma\mathbb{EP}(ch'_i,y_0)}$$

$$\frac{\partial\log Z_{\mathbf{CH}}(i,y_0,\gamma)}{\partial Q(ch_{i0}|y_0)} =$$
$$\frac{1}{Z_{\mathbf{CH}}(i,y_0,\gamma)}\sum_{ch'_i}\frac{\partial e^{(1-\gamma)\log Q(ch'_i)+\gamma\mathbb{EP}(ch'_i,y_0)}}{\partial Q(ch_{i0}|y_0)} =$$
$$\frac{1}{Z_{\mathbf{CH}}(i,y_0,\gamma)}\sum_{ch'_i}e^{(1-\gamma)\log Q(ch'_i)+\gamma\mathbb{EP}(ch'_i,y_0)}\frac{\partial((1-\gamma)\log Q(ch'_i)+\gamma\mathbb{EP}(ch'_i,y_0))}{\partial Q(ch_{i0}|y_0)} =$$
$$\frac{1}{Z_{\mathbf{CH}}(i,y_0,\gamma)}\sum_{ch'_i}Q(ch'_i)^{1-\gamma}\exp(\mathbb{EP}(ch'_i,y_0)^{\gamma})$$
$$\frac{\partial((1-\gamma)\log Q(ch'_i)+\gamma\mathbb{EP}(ch'_i,y_0))}{\partial Q(ch_{i0}|y_0)} =$$
$$\sum_{ch'_i}Q(ch'_i|y_0)\frac{\partial((1-\gamma)\log Q(ch'_i)+\gamma\mathbb{EP}(ch'_i,y_0))}{\partial Q(ch_{i0}|y_0)} =$$
$$Q(ch_{i0}|y_0)\frac{\partial((1-\gamma)\log Q(ch_{i0})+\gamma\mathbb{EP}(ch_{i0},y_0))}{\partial Q(ch_{i0}|y_0)}$$

Let us write $\log Z_{\mathbf{T}}(i,y_0,\gamma)$

$$\log Z_{\mathbf{T}}(i,y_0,\gamma)=\log\sum_{t'_i}e^{(1-\gamma)\log Q(t'_i)+\gamma\mathbb{EP}(t'_i,y_0)}$$

$$\frac{\partial\log Z_{\mathbf{T}}(i,y_0,\gamma)}{\partial Q(t_{i0}|y_0)} =$$
$$\frac{1}{Z_{\mathbf{T}}(i,y_0,\gamma)}\sum_{t'_i}\frac{\partial e^{(1-\gamma)\log Q(t'_i)+\gamma\mathbb{EP}(t'_i,y_0)}}{\partial Q(t_{i0}|y_0)} =$$
$$\frac{1}{Z_{\mathbf{T}}(i,y_0,\gamma)}\sum_{t'_i}e^{(1-\gamma)\log Q(t'_i)+\gamma\mathbb{EP}(t'_i,y_0)}\frac{\partial((1-\gamma)\log Q(t'_i)+\gamma\mathbb{EP}(t'_i,y_0))}{\partial Q(t_{i0}|y_0)} =$$

$$\frac{1}{Z_{\mathbf{T}}(i,y_0,\gamma)}\sum_{t'_i} Q(t'_i)^{1-\gamma}\exp(\mathbb{EP}(t'_i,y_0)^{\gamma})\frac{\partial((1-\gamma)\log Q(t'_i)+\gamma\mathbb{EP}(t'_i,y_0))}{\partial Q(t_{i0}|y_0)} =$$
$$\sum_{t'_i} Q(t'_i|y_0)\frac{\partial((1-\gamma)\log Q(t'_i)+\gamma\mathbb{EP}(t'_i,y_0))}{\partial Q(t_{i0}|y_0)} =$$
$$Q(t_{i0}|y_0)\frac{\partial((1-\gamma)\log Q(t_{i0})+\gamma\mathbb{EP}(t_{i0},y_0))}{\partial Q(t_{i0}|y_0)}$$

Thus, using the results obtained in Equations 2 and 2, the derivative of the G functions with respect to $Q(ch_i|y)$ and $Q(t_i|y)$ are

$$\frac{\partial G_{ch_i,y}(Q,\gamma)}{\partial Q(ch_i|y)} =$$
$$-\frac{1}{Q(ch_i|y)} + \frac{\partial((1-\gamma)\log Q(ch_i)+\gamma\mathbb{EP}(ch_i,y))}{\partial Q(ch_i|y)} -$$
$$Q(ch_i|y)\frac{\partial((1-\gamma)\log Q(ch_i)+\gamma\mathbb{EP}(ch_i,y))}{\partial Q(ch_i|y)} =$$
$$-\frac{1}{Q(ch_i|y)} + (1-Q(ch_i|y))\frac{\partial((1-\gamma)\log Q(ch_i)+\gamma\mathbb{EP}(ch_i,y))}{\partial Q(ch_i|y)} =$$
$$-\frac{1}{Q(ch_i|y)} + Q(y)(1-Q(ch_i|y))\left(\frac{(1-\gamma)}{Q(ch_i)} + \gamma\mathbb{E}_{Q(\mathbf{CH}|ch_i,\mathbf{pa}_{ch_i})}[\mathcal{D}(y_0, ch_{t(i,y),ch_{i0}}, tb(\mathbf{pa}_{ch_{t(i,y_0)}}))]\right)$$

$$\frac{\partial G_{t_i,y}(Q,\gamma)}{\partial Q(t_i|y)} =$$
$$-\frac{1}{Q(t_i|y)} + \frac{\partial((1-\gamma)\log Q(t_i)+\gamma\mathbb{EP}(t_i,y))}{\partial Q(t_i|y)} - Q(t_i|y)\frac{\partial((1-\gamma)\log Q(t_i)+\gamma\mathbb{EP}(t_i,y))}{\partial Q(t_i|y)} =$$
$$-\frac{1}{Q(t_i|y)} + (1-Q(t_i|y))\frac{\partial((1-\gamma)\log Q(t_i)+\gamma\mathbb{EP}(t_i,y))}{\partial Q(t_i|y)} =$$
$$-\frac{1}{Q(t_i|y)} + Q(y)(1-Q(t_i|y))\left(\frac{(1-\gamma)}{Q(t_i)} + \gamma\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|t_i,y)}[\mathcal{F}(y_0, ech(t_{i0}), et(t_{i0}))]\right)$$

Overall, the derivative of G with respect to ch_i and t_i are:

$$\frac{\partial G_{ch_i,y}(Q,\gamma)}{\partial Q(ch_i|y)} =$$
$$-\frac{1}{Q(ch_i|y)} + Q(y)(1-Q(ch_i|y))\left(\frac{1-\gamma}{Q(ch_i)} + \gamma\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|ch_i,y_0)}[\mathcal{D}(y_0, ch_{t(i,y),ch_{i0}}, tb(\mathbf{pa}_{ch_{t(i,y_0)}}))]\right)$$

$$\frac{\partial G_{t_i,y}(Q,\gamma)}{\partial Q(t_i|y)} = -\frac{1}{Q(t_i|y)} + Q(y)(1-Q(t_i|y))\left(\frac{1-\gamma}{Q(t_i)} + \gamma\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|t_i,y_0)}[\mathcal{F}(y_0, ech(t_{i0}), et(t_{i0}))]\right)$$

□

Theorem 2 (Derivatives of G with respect to γ).

$$\frac{\partial G_{ch_i,y}(Q,\gamma)}{\partial\gamma} = -\log Q(ch_i) + \mathbb{EP}'(ch_i,y) - \mathbb{E}_{Q(ch'_i|y)}[\mathbb{EP}'(ch'_i,y)] - \log Q(ch'_i)]$$

where

$$\mathbb{EP}'(ch_i, y) =$$
$$\prod_{t_j \in body(r(i))} Q(t_j|y) \log \theta_{Hd_{r(k)}=ch_i|\mathbf{pa}_{ch_k}}[y] 1\{body(\mathbf{pa}_{ch_i}) = true\} +$$
$$\delta(\sum_{t_j \in \mathbf{pa}^{\mathbf{T}}_{ch_i}} \prod_{t_j \in \mathbf{pa}^{\mathbf{T}}_{ch_i}} Q(t_j|y) 1\{body(\mathbf{pa}_{ch_i}) = false, ch_i \neq null\} +$$
$$R(i, \mathbf{ch}, y) + 1\{ch_i = x_j[y], val(ch_i)[y] = false\} + Q(T_j = false|y) 1\{ch_i = t_j, t_j = false\})$$

$$R(i, \mathbf{ch}, y) =$$
$$\sum_{j \in p(i), x_j[y]=true, ch_i \neq x_j[y]} \prod_{ch_s \in \mathbf{pa}_{x_j}, s \neq i} Q(ch_s \neq x_j[y]|y) +$$
$$\sum_{j \in p(i), t_j=true, ch_i \neq t_j} \prod_{ch_s \in \mathbf{pa}_{t_j}, s \neq i} Q(ch_s \neq t_j|y) Q(T_j = true|y)$$

$$\frac{\partial G_{t_i,y}(Q,\gamma)}{\partial \gamma} = -\log Q(t_i) + \mathbb{EP}'(t_i, y) - E_{Q(t'_i|y)}[\mathbb{EP}'(t'_i, y) - \log Q(t'_i)]$$

where

$$\mathbb{EP}'(t_i, y) =$$
$$\sum_{k \in bb(t_i,y)} \sum_{ch_k \neq null} Q(ch_k|y) \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y) \log \theta_{Hd_{r(k)}=ch_k|true} +$$
$$\delta(S(i, t, y) + \prod_{Ch_s \in \mathbf{pa}_{t_i}} Q(Ch_s \neq t_i|y)\{t_i = true\}\delta +$$
$$\left(1 - \prod_{Ch_s \in \mathbf{pa}_{t_i}} Q(Ch_s \neq t_i|y)\right)\{t_i = false\}\delta$$

$$S(i, \mathbf{t}, y) =$$
$$\sum_{k \in bb(t_i,y)} Q(Ch_k = null|y) \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y) +$$
$$\sum_{k \in bb(t_i,y)} Q(Ch_k \neq null|y) \left(1 - \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y)\right) +$$
$$\sum_{k, body(k)[y]=true, \overline{t_i} \in body(k)} Q(Ch_k \neq null|y) +$$
$$\sum_{k, body(k)[y]=false, t_i \in body(k)} Q(Ch_k \neq null|y)$$

and $\delta \approx \log 0$ *(e.g.* $\delta = -/10$*),*

Proof.

$$\frac{\partial \log Z_{\mathbf{CH}}(i, y, \gamma)}{\partial \gamma} =$$

$$\frac{1}{Z_{\mathbf{CH}}(i,y,\gamma)}\sum_{ch'_i} e^{(1-\gamma)\log Q(ch'_i)+\gamma\mathbb{EP}(ch'_i,y)}\frac{\partial((1-\gamma)\log Q(ch'_i)+\gamma\mathbb{EP}(ch'_i,y))}{\partial\gamma} =$$
$$\frac{1}{Z_{\mathbf{CH}}(i,y,\gamma)}\sum_{ch'_i} Q(ch'_i)^{1-\gamma}\mathbb{EP}(ch'_i,y)^{\gamma}(-\log Q(ch'_i)+\mathbb{EP}(ch'_i,y)) =$$
$$\sum_{ch'_i} Q(ch'_i|y)(\mathbb{EP}(ch'_i,y)-\log Q(ch'_i)) =$$
$$\mathbb{E}_{Q(ch_i|y)}[\mathbb{EP}(ch_i,y)-\log Q(ch_i)]$$

$$\frac{\partial G_{ch_i,y}(Q,\gamma)}{\partial\gamma} = -\log Q(ch_i)+\mathbb{EP}(ch_i,y)-\mathbb{E}_{Q(ch'_i|y)}[\mathbb{EP}(ch'_i,y)-\log Q(ch'_i)]$$

$$\frac{\partial\log Z_{\mathbf{T}}(i,y,\gamma)}{\partial\gamma} =$$
$$\frac{1}{Z_{\mathbf{T}}(i,y,\gamma)}\sum_{t'_i} e^{(1-\gamma)\log Q(t'_i)+\gamma\mathbb{EP}(t'_i,y)}\frac{\partial((1-\gamma)\log Q(t'_i)+\gamma\mathbb{EP}(t'_i,y))}{\partial\gamma} =$$
$$\frac{1}{Z_{\mathbf{T}}(i,y,\gamma)}\sum_{t'_i} Q(t'_i)^{1-\gamma}\mathbb{EP}(t'_i,y)^{\gamma}(-\log Q(t'_i)+\mathbb{EP}(t'_i,y)) =$$
$$\sum_{t'_i} Q(t'_i|y)(\mathbb{EP}(t'_i,y)-\log Q(t'_i)) =$$
$$\mathbb{E}_{Q(t_i|y)}[\mathbb{EP}(t_i,y)-\log Q(t_i)]$$
$$\frac{\partial G_{t_i,y}(Q,\gamma)}{\partial\gamma} = -\log Q(t_i)+\mathbb{EP}(t_i,y)-\mathbb{E}_{Q(t'_i|y)}[\mathbb{EP}(t'_i,y)-\log Q(t'_i)]$$

Let us now see how to compute $\mathbb{EP}(ch_i,y)$

$$\mathbb{EP}(ch_i,y) = \mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|ch_i,y)}[\log P(\mathbf{x}[y],\mathbf{T},\mathbf{CH})] =$$
$$\sum_{\mathbf{ch},\mathbf{ch}[i]\neq ch_i,\mathbf{t}} Q(\mathbf{ch},\mathbf{t}|ch_i,y)\left(\sum_k \log\theta_{Hd_{r(k)}=ch_k|\mathbf{pa}_{ch_k}}[y]+\sum_j \log\theta_{x_j|\mathbf{pa}_{x_j}}[y]+\sum_k \log\theta_{t_k|\mathbf{pa}_{t_k}}\right) =$$
$$\sum_k \sum_{\mathbf{ch},\mathbf{ch}[i]\neq ch_i,t} Q(\mathbf{ch},\mathbf{t}|ch_i,y)\log\theta_{Hd_{r(k)}=ch_k|\mathbf{pa}_{ch_k}}[y] +$$
$$\sum_j \sum_{\mathbf{ch},\mathbf{ch}[i]\neq ch_i,t} Q(\mathbf{ch},\mathbf{t}|ch_i,y)\log\theta_{x_j|\mathbf{pa}_{x_j}}[y] +$$
$$\sum_k \sum_{\mathbf{ch},\mathbf{ch}[i]\neq ch_i,t} Q(\mathbf{ch},\mathbf{t}|ch_i,y)\log\theta_{t_k|\mathbf{pa}_{t_k}} =$$
$$\sum_{t\in pa_{ch_i}} \prod_{t_j\in\mathbf{pa}_{ch_i}} Q(t_j|y)\log\theta_{Hd_{r(i)}=ch_i|\mathbf{pa}_{ch_i}}[y] +$$
$$\sum_{k\neq i} \sum_{ch_k,t\in\mathbf{pa}_{ch_k}} Q(ch_k,t|ch_i,y)\log\theta_{Hd_{r(k)}=ch_k|\mathbf{pa}_{ch_k}}[y] +$$
$$\sum_j \sum_{\mathbf{pa}_{x_j}\setminus\{ch_i\}} Q(ch_{x_j}|ch_i,y)\log\theta_{x_j|\mathbf{pa}_{x_j}}[y] +$$
$$\sum_k \sum_{\mathbf{pa}_{t_k}\setminus\{ch_i\}} Q(ch_{t_k}|ch_i,y)\log\theta_{t_k|\mathbf{pa}_{t_k}} =$$

$$\sum_{t \in \mathbf{pa}_{ch_i}} \prod_{t_j \in \mathbf{pa}_{ch_i}} Q(t_j|y) \log \theta_{Hd_{r(i)}=ch_i|\mathbf{pa}_{ch_i}}[y] +$$
$$\sum_{k \neq i} \sum_{ch_k, t \in \mathbf{pa}_{ch_k}} Q(ch_k, t|ch_i, y) \log \theta_{Hd_{r(k)}=ch_k|\mathbf{pa}_{ch_k}}[y] +$$
$$\sum_{j} \sum_{\mathbf{pa}_{x_j} \setminus \{ch_i\}} \prod_{ch_s \in \mathbf{pa}_{x_j}, s \neq i} Q(ch_s|y) \log \theta_{x_j|\mathbf{pa}_{x_j}}[y] + \sum_{k} \sum_{\mathbf{pa}_{t_k} \setminus \{ch_i\}} Q(ch_{t_k}|ch_i, y) \log \theta_{t_k|\mathbf{pa}_{t_k}}$$

So $\mathbb{EP}(ch_i, y)$ can be computed without inference. Let us write $\mathbb{EP}(ch_i, y)$ as

$$\mathbb{EP}(ch_i, y) = \mathbb{EP}_1(ch_i, y) + \mathbb{EP}_2(y)$$

where $\mathbb{EP}_2(y)$ does not depend on ch_i and

$$\mathbb{EP}_1(ch_i, y) =$$
$$\sum_{t \in \mathbf{pa}_{ch_i}} \prod_{t_j \in \mathbf{pa}_{ch_i}} Q(t_j|y) \log \theta_{Hd_{r(k)}=ch_i|\mathbf{pa}_{ch_k}}[y] +$$
$$\sum_{j \in p(i)} \sum_{\mathbf{pa}_{x_j} \setminus \{ch_i\}} \prod_{ch_s \in \mathbf{pa}_{x_j}, s \neq i} Q(ch_s|y) \log \theta_{x_j|\mathbf{pa}_{x_j}}[y] +$$
$$\sum_{j \in p(i)} \sum_{\mathbf{pa}_{t_j} \setminus \{ch_i\}, t_j} \prod_{ch_s \in \mathbf{pa}_{x_j}, s \neq i} Q(ch_s|y) Q(t_j|y) \log \theta_{t_j|\mathbf{pa}_{x_j}}[y]$$

where $p(i) = \{j | ch_i \in \mathbf{pa}_{x_j}\} = val(ch_i) \setminus \{null\}$.

$$\mathbb{EP}_1(ch_i, y) =$$
$$\prod_{t_j \in body(r(i))} Q(t_j|y) \log \theta_{Hd_{r(i)}=ch_i|\mathbf{pa}_{ch_i}}[y] 1\{body(\mathbf{pa}_{ch_i}) = true\} +$$
$$\sum_{t_j \in \mathbf{pa}^T_{ch_i}} \prod_{t_j \in \mathbf{pa}^{\mathbf{T}}_{ch_i}} Q(t_j|y) 1\{body(\mathbf{pa}_{ch_i}) = false, ch_i \neq null\} \delta +$$
$$\sum_{j \in p(i), x_j[y]=true} \sum_{\mathbf{pa}_{x_j} \setminus \{ch_i\}, x_j[y] \notin val(\mathbf{pa}_{x_j})} \prod_{ch_s \in \mathbf{pa}_{x_j}, s \neq i} Q(ch_s|y) \delta +$$
$$\sum_{j \in p(i), x_j[y]=false} \sum_{\mathbf{pa}_{x_j} \setminus \{ch_i\}, x_j[y] \in val(\mathbf{pa}_{x_j})} \prod_{ch_s \in \mathbf{pa}_{x_j}, s \neq i} Q(ch_s|y) \delta +$$
$$\sum_{j \in p(i), t_j=true} \sum_{\mathbf{pa}_{t_j} \setminus \{ch_i\}, t_j \notin val(\mathbf{pa}_{t_j})} \prod_{ch_s \in \mathbf{pa}_{x_j}, s \neq i} Q(ch_s|y) Q(T_j = true|y) \delta +$$
$$\sum_{j \in p(i), t_j=false} \sum_{\mathbf{pa}_{t_j} \setminus \{ch_i\}, t_j \in val(\mathbf{pa}_{t_j})} \prod_{ch_s \in \mathbf{pa}_{x_j}, s \neq i} Q(ch_s|y) Q(T_j = false|y) \delta =$$
$$\prod_{t_j \in body(r(i))} Q(t_j|y) \log \theta_{Hd_{r(i)}=ch_i|\mathbf{pa}_{ch_i}}[y] 1\{body(\mathbf{pa}_{ch_i}) = true\} +$$
$$\delta(\sum_{t_j \in \mathbf{pa}^{\mathbf{T}}_{ch_i}} \prod_{t_j \in \mathbf{pa}^{\mathbf{T}}_{ch_i}} Q(t_j|y) 1\{body(\mathbf{pa}_{ch_i}) = false, ch_i \neq null\} +$$
$$\sum_{j \in p(i), x_j[y]=true} \sum_{\mathbf{pa}_{x_j} \setminus \{ch_i\}, x_j[y] \notin val(\mathbf{pa}_{x_j})} \prod_{ch_s \in \mathbf{pa}_{x_j}, s \neq i} Q(ch_s|y) +$$
$$\sum_{j \in p(i), x_j[y]=false} \sum_{\mathbf{pa}_{x_j} \setminus \{ch_i\}, x_j[y] \in val(\mathbf{pa}_{x_j})} \prod_{ch_s \in \mathbf{pa}_{x_j}, s \neq i} Q(ch_s|y) +$$

$$\sum_{j\in p(i),t_j=true}\sum_{\mathbf{pa}_{t_j}\setminus\{ch_i\},t_j\notin val(\mathbf{pa}_{t_j})}\prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} Q(ch_s|y)Q(T_j=true|y)+$$

$$\sum_{j\in p(i),t_j=false}\sum_{\mathbf{pa}_{t_j}\setminus\{ch_i\},t_j\in val(\mathbf{pa}_{t_j})}\prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} Q(ch_s|y)Q(T_j=false|y))=$$

$$\prod_{t_j\in body(r(i))} Q(t_j|y)\log\theta_{Hd_{r(i)}=ch_i|\mathbf{pa}_{ch_i}}[y]1\{body(\mathbf{pa}_{ch_i})=true\}+$$

$$\delta(\sum_{t_j\in\mathbf{pa}^{\mathbf{T}}_{ch_i}}\prod_{t_j\in\mathbf{pa}^{\mathbf{T}}_{ch_i}} Q(t_j|y)1\{body(\mathbf{pa}_{ch_i})=false, ch_i\neq null\}+$$

$$\sum_{j\in p(i),x_j[y]=true}\sum_{\mathbf{pa}_{x_j}\setminus\{ch_i\},x_j[y]\notin val(\mathbf{pa}_{x_j})\setminus\{ch_i\}}\prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} Q(ch_s|y)1\{ch_i\neq x_j\}+$$

$$\sum_{j\in p(i),x_j[y]=false}\sum_{\mathbf{pa}_{x_j}\setminus\{ch_i\},x_j[y]\in val(\mathbf{pa}_{x_j})\setminus\{ch_i\}}\prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} Q(ch_s|y)+$$

$$\sum_{j\in p(i),x_j[y]=false}\sum_{\mathbf{pa}_{x_j}\setminus\{ch_i\}}\prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} Q(ch_s|y)1\{ch_i=x_j[y]\}+$$

$$\sum_{j\in p(i),t_j=true}\sum_{\mathbf{pa}_{t_j}\setminus\{ch_i\},t_j\notin val(\mathbf{pa}_{t_j})\setminus\{ch_i\}}\prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} Q(ch_s|y)Q(T_j=true|y)1\{ch_i\neq t_j\}+$$

$$\sum_{j\in p(i),t_j=false}\sum_{\mathbf{pa}_{t_j}\setminus\{ch_i\},t_j\in val(\mathbf{pa}_{t_j}\setminus\{ch_i\})}\prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} Q(ch_s|y)Q(T_j=false|y))+$$

$$\sum_{j\in p(i),t_j=false}\sum_{\mathbf{pa}_{t_j}\setminus\{ch_i\}}\prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} Q(ch_s|y)Q(T_j=false|y)1\{ch_i=t_j\}=$$

$$\prod_{t_j\in body(r(i))} Q(t_j|y)\log\theta_{Hd_{r(i)}=ch_i|\mathbf{pa}_{ch_i}}[y]1\{body(\mathbf{pa}_{ch_i})=true\}+$$

$$\delta(\sum_{t_j\in\mathbf{pa}^{\mathbf{T}}_{ch_i}}\prod_{t_j\in\mathbf{pa}^{\mathbf{T}}_{ch_i}} Q(t_j|y)1\{body(\mathbf{pa}_{ch_i})=false, ch_i\neq null\}+$$

$$\sum_{j\in p(i),x_j[y]=true}\sum_{\mathbf{pa}_{x_j}\setminus\{ch_i\},x_j[y]\notin val(\mathbf{pa}_{x_j})\setminus\{ch_i\}}\prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} Q(ch_s|y)1\{ch_i\neq x_j\}+$$

$$\sum_{j\in p(i),x_j[y]=false}\sum_{\mathbf{pa}_{x_j}\setminus\{ch_i\},x_j[y]\in val(\mathbf{pa}_{x_j})\setminus\{ch_i\}}\prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} Q(ch_s|y)+$$

$$\sum_{j\in p(i),x_j[y]=false} 1\{ch_i=x_j[y]\}+$$

$$\sum_{j\in p(i),t_j=true}\sum_{\mathbf{pa}_{t_j}\setminus\{ch_i\},t_j\notin val(\mathbf{pa}_{t_j})\setminus\{ch_i\}}\prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} Q(ch_s|y)Q(T_j=true|y)1\{ch_i\neq t_j\}+$$

$$\sum_{j\in p(i),t_j=false}\sum_{\mathbf{pa}_{t_j}\setminus\{ch_i\},t_j\in val(\mathbf{pa}_{t_j}\setminus\{ch_i\})}\prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} Q(ch_s|y)Q(T_j=false|y))+$$

$$\sum_{j\in p(i),t_j=false} Q(T_j=false|y)1\{ch_i=t_j\})=$$

$$\prod_{t_j\in body(r(i))} Q(t_j|y)\log\theta_{Hd_{r(i)}=ch_i|\mathbf{pa}_{ch_i}}[y]1\{body(\mathbf{pa}_{ch_i})=true\}+$$

$$
\begin{aligned}
&\delta(\sum_{t_j\in \mathbf{pa}^{\mathbf{T}}_{ch_i}} \prod_{t_j\in \mathbf{pa}^{\mathbf{T}}_{ch_i}} Q(t_j|y)1\{body(\mathbf{pa}_{ch_i}) = false, ch_i \neq null\} + \\
&\sum_{j\in p(i),x_j[y]=true} \sum_{\mathbf{pa}_{x_j}\backslash\{ch_i\},x_j[y]\notin val(\mathbf{pa}_{x_j})\backslash\{ch_i\}} \prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} Q(ch_s|y)1\{ch_i\neq x_j\} + \\
&\sum_{j\in p(i),x_j[y]=false} \sum_{\mathbf{pa}_{x_j}\backslash\{ch_i\},x_j[y]\in val(\mathbf{pa}_{x_j})\backslash\{ch_i\}} \prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} Q(ch_s|y) + \\
&\sum_{j\in p(i),t_j=true} \sum_{\mathbf{pa}_{t_j}\backslash\{ch_i\},t_j\notin val(\mathbf{pa}_{t_j})\backslash\{ch_i\}} \prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} Q(ch_s|y)Q(T_j=true|y)1\{ch_i\neq t_j\} + \\
&\sum_{j\in p(i),t_j=false} \sum_{\mathbf{pa}_{t_j}\backslash\{ch_i\},t_j\in val(\mathbf{pa}_{t_j}\backslash\{ch_i\})} \prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} Q(ch_s|y)Q(T_j=false|y)) + \\
&1\{ch_i = x_j[y], val(ch_i)[y] = false\} + \\
&Q(T_j = false|y)1\{ch_i = t_j, val(ch_i) = t_j = false\})
\end{aligned}
$$

where δ is used to approximate $\log 0$ (e.g. $\delta = -10$). Let

$$
\begin{aligned}
&\mathbb{EP}'(ch_i, y) = \\
&\prod_{t_j\in body(r(i))} Q(t_j|y)\log\theta_{Hd_{r(k)}=ch_i|\mathbf{pa}_{ch_k}}[y]1\{body(\mathbf{pa}_{ch_i}) = true\} + \\
&\delta(\sum_{t_j\in \mathbf{pa}^{\mathbf{T}}_{ch_i}} \prod_{t_j\in \mathbf{pa}^{\mathbf{T}}_{ch_i}} Q(t_j|y)1\{body(\mathbf{pa}_{ch_i}) = false, ch_i \neq null\} + \\
&\sum_{j\in p(i),x_j[y]=true} \sum_{\mathbf{pa}_{x_j}\backslash\{ch_i\},x_j[y]\notin val(\mathbf{pa}_{x_j})\backslash\{ch_i\}} \prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} Q(ch_s|y)1\{ch_i\neq x_j\} + \\
&\sum_{j\in p(i),t_j=true} \sum_{\mathbf{pa}_{t_j}\backslash\{ch_i\},t_j\notin val(\mathbf{pa}_{t_j})\backslash\{ch_i\}} \prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} Q(ch_s|y)Q(T_j=true|y)1\{ch_i\neq t_j\} + \\
&1\{ch_i = x_j[y], val(ch_i)[y] = false\} + \\
&Q(T_j = false|y)1\{ch_i = t_j, val(ch_i) = t_j = false\}) = \\
&\prod_{t_j\in body(r(i))} Q(t_j|y)\log\theta_{Hd_{r(k)}=ch_i|\mathbf{pa}_{ch_k}}[y]1\{body(\mathbf{pa}_{ch_i}) = true\} + \\
&\delta(\sum_{t_j\in \mathbf{pa}^{\mathbf{T}}_{ch_i}} \prod_{t_j\in \mathbf{pa}^{\mathbf{T}}_{ch_i}} Q(t_j|y)1\{body(\mathbf{pa}_{ch_i}) = false, ch_i \neq null\} + \\
&\sum_{j\in p(i),x_j[y]=true} \prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i} \sum_{\mathbf{pa}_{x_j}\backslash\{ch_i\},x_j[y]\notin val(\mathbf{pa}_{x_j})\backslash\{ch_i\}} Q(ch_s|y)1\{ch_i\neq x_j\} + \\
&\sum_{j\in p(i),t_j=true} \prod_{ch_s\in\mathbf{pa}_{t_j},s\neq i} \sum_{\mathbf{pa}_{t_j}\backslash\{ch_i\},t_j\notin val(\mathbf{pa}_{t_j})\backslash\{ch_i\}} Q(ch_s|y)Q(T_j=true|y)1\{ch_i\neq t_j\} + \\
&1\{ch_i = x_j[y], val(ch_i)[y] = false\} + \\
&Q(T_j = false|y)1\{ch_i = t_j, val(ch_i) = t_j = false\}) = \\
&\prod_{t_j\in body(r(i))} Q(t_j|y)\log\theta_{Hd_{r(k)}=ch_i|\mathbf{pa}_{ch_k}}[y]1\{body(\mathbf{pa}_{ch_i}) = true\} + \\
&\delta(\sum_{t_j\in \mathbf{pa}^{\mathbf{T}}_{ch_i}} \prod_{t_j\in \mathbf{pa}^{\mathbf{T}}_{ch_i}} Q(t_j|y)1\{body(\mathbf{pa}_{ch_i}) = false, ch_i \neq null\} +
\end{aligned}
$$

$$\sum_{j\in p(i),x_j[y]=true,ch_i\neq x_j[y]}\prod_{ch_s\in \mathbf{pa}_{x_j},s\neq i}Q(ch_s\neq x_j[y]|y)+$$
$$\sum_{j\in p(i),t_j=true,ch_i\neq t_j}\prod_{ch_s\in \mathbf{pa}_{t_j},s\neq i}Q(ch_s\neq t_j|y)Q(T_j=true|y)+$$
$$1\{ch_i=x_j[y],val(ch_i)[y]=false\}+$$
$$Q(T_j=false|y)1\{ch_i=t_j,val(ch_i)=t_j=false\})$$

and

$$\mathbb{EP}_3(y)=$$
$$\sum_{j\in p(i),x_j[y]=false}\sum_{\mathbf{pa}_{x_j}\setminus\{ch_i\},x_j[y]\in val(\mathbf{pa}_{x_j})\setminus\{ch_i\}}\prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i}Q(ch_s|y)+$$
$$\sum_{j\in p(i),t_j=false}\sum_{\mathbf{pa}_{t_j}\setminus\{ch_i\},t_j\in val(\mathbf{pa}_{t_j}\setminus\{ch_i\})}\prod_{ch_s\in\mathbf{pa}_{x_j},s\neq i}Q(ch_s|y)Q(T_j=false|y))$$

where $\mathbb{EP}_3(y)$ does not depend on ch_i $\mathbb{EP}(ch_i,y)=\mathbb{EP}'(ch_i,y)+\mathbb{EP}_2(y)+\mathbb{EP}_3(y)=\mathbb{EP}'(ch_i,y)+\mathbb{EP}''(y)$ and $\mathbb{EP}''(y)$ does not depend on ch_i Let

$$R(i,\mathbf{ch},y)=$$
$$\sum_{j\in p(i),x_j[y]=true,ch_i\neq x_j[y]}\prod_{ch_s\in \mathbf{pa}_{x_j},s\neq i}Q(ch_s\neq x_j[y]|y)+$$
$$\sum_{j\in p(i),t_j=true,ch_i\neq t_j}\prod_{ch_s\in \mathbf{pa}_{t_j},s\neq i}Q(ch_s\neq t_j|y)Q(T_j=true|y)$$

then

$$\mathbb{EP}'(ch_i,y)=$$
$$\prod_{t_j\in body(r(i))}Q(t_j|y)\log\theta_{Hd_{r(k)}=ch_i|\mathbf{pa}_{ch_k}}[y]1\{body(\mathbf{pa}_{ch_i})=true\}+$$
$$\delta(\sum_{t_j\in\mathbf{pa}^{\mathbf{T}}_{ch_i}}\prod_{t_j\in\mathbf{pa}^{\mathbf{T}}_{ch_i}}Q(t_j|y)1\{body(\mathbf{pa}_{ch_i})=false,ch_i\neq null\}+$$
$$R(i,\mathbf{ch},y)+1\{ch_i=x_j[y],val(ch_i)[y]=false\}+$$
$$Q(T_j=false|y)1\{ch_i=t_j,val(ch_i)=t_j=false\})$$

So

$$\frac{\partial G_{ch_i,y}(Q,\gamma)}{\partial\gamma}=$$
$$-\log Q(ch_i)+\mathbb{EP}'(ch_i,y)+\mathbb{EP}''(y)-\mathbb{E}_{Q(ch'_i|y)}[\mathbb{EP}'(ch'_i,y)+\mathbb{EP}''(y)-\log Q(ch'_i)]=$$
$$-\log Q(ch_i)+\mathbb{EP}'(ch_i,y)+\mathbb{EP}''(y)-\mathbb{EP}''(y)-\mathbb{E}_{Q(ch'_i|y)}[\mathbb{EP}'(ch'_i,y)-\log Q(ch'_i)]=$$
$$-\log Q(ch_i)+\mathbb{EP}'(ch_i,y)-\mathbb{E}_{Q(ch'_i|y)}[\mathbb{EP}'(ch'_i,y)-\log Q(ch'_i)]$$

Let us now see how to compute $\mathbb{EP}(t_i,y)$

$$\mathbb{EP}(t_i,y)=$$
$$\mathbb{E}_{Q(\mathbf{CH},\mathbf{T}|t_i,y)}[\log P(\mathbf{x}[y],\mathbf{T},\mathbf{CH})]=$$

$$\sum_{ch,t\neq t_i} Q(ch,t|t_i,y)(\sum_{k\in b(t_i,y)} \log\theta_{Hd_{r(k)}=ch_k|\mathbf{pa}_{ch_k}} + \sum_{k\notin b(t_i,y)} \log\theta_{Hd_{r(i)}=ch_k|\mathbf{pa}_{ch_k}} +$$
$$\sum_j \log\theta_{x_j|\mathbf{pa}_{x_j}}[y] + \sum_{k\neq i}\log\theta_{t_k|\mathbf{pa}_{t_k}} + \log\theta_{t_i|\mathbf{pa}_{t_i}}) =$$
$$\sum_{k\in bb(t_i,y)} \sum_{ch,t\backslash t_i} Q(\mathbf{ch},\mathbf{t}|t_i,y)\log\theta_{Hd_{r(k)}=ch_k|\mathbf{pa}_{ch_k}} +$$
$$\sum_{k\notin bb(t_i,y)} \sum_{\mathbf{ch},\mathbf{t}\neq t_i} Q(\mathbf{ch},\mathbf{t}|t_i,y)\log\theta_{Hd_{r(i)}=ch_k|\mathbf{pa}_{ch_k}} +$$
$$\sum_j \sum_{\mathbf{ch},\mathbf{t}\neq t_i} Q(\mathbf{ch},\mathbf{t}|t_i,y)\log\theta_{x_j|\mathbf{pa}_{x_j}} + \sum_{k\neq i}\sum_{\mathbf{ch},\mathbf{t}\neq t_i} Q(\mathbf{ch},\mathbf{t}|t_i,y)\log\theta_{t_k|\mathbf{pa}_{t_k}} +$$
$$\sum_{\mathbf{ch},\mathbf{t}\backslash t_i} Q(\mathbf{ch},\mathbf{t}|t_i,y)\log\theta_{t_i|\mathbf{pa}_{t_i}} =$$
$$\sum_{k\in bb(t_i,y)} \sum_{ch_k} Q(ch_k|y) \sum_{t\in body(k),t\neq t_i} \prod_{t_j\in body(r(k)),t_j\neq t_i} Q(t_j|y)\log\theta_{Hd_{r(k)}=ch_k|\mathbf{pa}_{ch_k}} +$$
$$\sum_{k\notin bb(t_i,y)} \sum_{ch_k} Q(ch_k|y) \sum_{t\in body(k),t\neq t_i} \prod_{t_j\in body(r(k)),t_j\neq t_i} Q(t_j|y)\log\theta_{Hd_{r(i)}=ch_k|\mathbf{pa}_{ch_k}} +$$
$$\sum_j \sum_{\mathbf{pa}_{x_j}} Q(\mathbf{pa}_{x_j}|y)\log\theta_{x_j|\mathbf{pa}_{x_j}} + \sum_{k\neq i}\sum_{\mathbf{pa}_{t_k}} Q(\mathbf{pa}_{t_k}|y)\log\theta_{t_k|\mathbf{pa}_{t_k}} + \sum_{\mathbf{pa}_{t_i}} Q(\mathbf{pa}_{t_i}|y)\log\theta_{t_i|\mathbf{pa}_{t_i}}$$

where $body(k)$ body of the rule for ch_k, $body(k)[y]$ the portion of body restricted to $\mathbf{X}$ variables taking the values $\mathbf{x}[y]$, $bb(t_i,y) = \{k|body(k)[y] = true, t_i \in body(k)\}$.

So $\mathbb{EP}(t_i,y)$ can be computed without inference.

$$\mathbb{EP}(t_i,y) = \mathbb{EP}_1(t_i,y) + \mathbb{EP}_2(y)$$

where $\mathbb{EP}_2(y)$ is

$$\mathbb{EP}_2(y) = \sum_j \sum_{\mathbf{pa}_{x_j}} Q(\mathbf{pa}_{x_j}|y)\log\theta_{x_j|\mathbf{pa}_{x_j}} + \sum_{k\neq i}\sum_{\mathbf{pa}_{t_k}} Q(\mathbf{pa}_{t_k}|y)\log\theta_{t_k|\mathbf{pa}_{t_k}}$$

and it does not depend on t_i and

$$\mathbb{EP}_1(t_i,y) =$$
$$\sum_{k\in bb(t_i,y)} \sum_{ch_k} Q(ch_k|y) \sum_{t\in body(k),t\neq t_i} \prod_{t_j\in body(k),t_j\neq t_i} Q(t_j|y)\log\theta_{Hd_{r(k)}=ch_k|\mathbf{pa}_{ch_k}} +$$
$$\sum_{k\notin bb(t_i,y)} \sum_{ch_k} Q(ch_k|y) \sum_{t\in body(k),t\neq t_i} \prod_{t_j\in body(k),t_j\neq t_i} Q(t_j|y)\log\theta_{Hd_{r(i)}=ch_k|\mathbf{pa}_{ch_k}} +$$
$$\sum_{\mathbf{pa}_{t_i}} Q(\mathbf{pa}_{t_i}|y)\log\theta_{t_i|\mathbf{pa}_{t_i}} =$$
$$\sum_{k\in bb(t_i,y)} \sum_{ch_k} Q(ch_k|y) \sum_{t\in body(k),t\neq t_i} \prod_{t_j\in body(k),t_j\neq t_i} Q(t_j|y)\log\theta_{Hd_{r(k)}=ch_k|\mathbf{pa}_{ch_k}} +$$
$$\sum_{k\notin bb(t_i,y),T_i\in body(k)} \sum_{ch_k} Q(ch_k|y) \sum_{t\in body(k),t\neq t_i} \prod_{t_j\in body(k),t_j\neq t_i} Q(t_j|y)\log\theta_{Hd_{r(i)}=ch_k|\mathbf{pa}_{ch_k}} +$$
$$\sum_{k\notin bb(t_i,y),T_i\notin body(k)} \sum_{ch_k} Q(ch_k|y) \sum_{t\in body(k),t\neq t_i} \prod_{t_j\in body(k),t_j\neq t_i} Q(t_j|y)\log\theta_{Hd_{r(i)}=ch_k|\mathbf{pa}_{ch_k}} +$$

$$\sum_{\mathbf{pa}_{t_i}} Q(\mathbf{pa}_{t_i}|y) \log \theta_{t_i|\mathbf{pa}_{t_i}}$$

So $\mathbb{EP}_1(t_i, y)$

$$\mathbb{EP}_1(t_i, y) = \mathbb{EP}'(t_i, y) + \mathbb{EP}_3(y)$$

where $\mathbb{EP}_3(y)$ is

$$\mathbb{EP}_3(y) = \sum_{k \notin bb(t_i,y), T_i \notin body(k)} \sum_{ch_k} Q(ch_k|y) \sum_{t \in body(k), t \neq t_i} \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y) \log \theta_{Hd_{r(i)}=ch_k|\mathbf{pa}_{ch_k}}$$

and it does not depend on t_i and

$$\begin{aligned}
&\mathbb{EP}'(t_i, y) = \\
&\sum_{k \in bb(t_i,y)} \sum_{ch_k} Q(ch_k|y) \sum_{t \in body(k), t \neq t_i} \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y) \log \theta_{Hd_{r(k)}=ch_k|\mathbf{pa}_{ch_k}} + \\
&\sum_{k \notin bb(t_i,y), T_i \in body(k)} \sum_{ch_k} Q(ch_k|y) \sum_{t \in body(k), t \neq t_i} \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y) \log \theta_{Hd_{r(k)}=ch_k|\mathbf{pa}_{ch_k}} + \\
&\sum_{\mathbf{pa}_{t_i}} Q(\mathbf{pa}_{t_i}|y) \log \theta_{t_i|\mathbf{pa}_{t_i}} = \\
&\sum_{k \in bb(t_i,y)} \sum_{ch_k \neq null} Q(ch_k|y) \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y) \log \theta_{Hd_{r(k)}=ch_k|true} + \\
&\sum_{k \in bb(t_i,y)} Q(Ch_k = null|y) \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y)\delta + \\
&\sum_{k \in bb(t_i,y)} \sum_{ch_k \neq null} Q(ch_k|y)(1 - \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y))\delta + \\
&\sum_{k \notin bb(t_i,y), T_i \in body(k)} \sum_{ch_k} Q(ch_k|y) \sum_{t \in body(k), t \neq t_i} \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y) \log \theta_{Hd_{r(k)}=ch_k|\mathbf{pa}_{ch_k}} + \\
&\sum_{pa_{t_i}, T_i \notin \mathbf{pa}_{t_i}} \prod_{ch_s \in \mathbf{pa}_{t_i}} Q(ch_s|y)\{t_i = true\}\delta + \\
&\sum_{\mathbf{pa}_{t_i}, T_i \in \mathbf{pa}_{t_i}} \prod_{ch_s \in \mathbf{pa}_{t_i}} Q(ch_s|y)\{t_i = false\}\delta = \\
&\sum_{k \in bb(t_i,y)} \sum_{ch_k \neq null} Q(ch_k|y) \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y) \log \theta_{Hd_{r(k)}=ch_k|true} + \\
&\sum_{k \in bb(t_i,y)} Q(Ch_k = null|y) \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y)\delta + \\
&\sum_{k \in bb(t_i,y)} Q(Ch_k \neq null|y)(1 - \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y))\delta + \\
&\sum_{k, body(k)[y]=true, \overline{t_i} \in body(k)} Q(Ch_k \neq null|y)\delta + \\
&\sum_{k, body(k)[y]=false, t_i \in body(k)} Q(Ch_k \neq null|y)\delta + \\
&\prod_{Ch_s \in \mathbf{pa}_{t_i}} Q(Ch_s \neq t_i|y)\{t_i = true\}\delta +
\end{aligned}$$

$$\left(1 - \prod_{Ch_s \in \mathbf{pa}_{t_i}} Q(Ch_s \neq t_i|y)\right) \{t_i = false\}\delta =$$

$$\sum_{k \in bb(t_i,y)} \sum_{ch_k \neq null} Q(ch_k|y) \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y) \log \theta_{Hd_{r(k)}=ch_k|true} +$$

$$\delta(\sum_{k \in bb(t_i,y)} Q(Ch_k = null|y) \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y) +$$

$$\sum_{k \in bb(t_i,y)} Q(Ch_k \neq null|y) \left(1 - \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y)\right) +$$

$$\sum_{k, body(k)[y]=true, \overline{t_i} \in body(k)} Q(Ch_k \neq null|y) +$$

$$\sum_{k, body(k)[y]=false, t_i \in body(k)} Q(Ch_k \neq null|y) +$$

$$\prod_{Ch_s \in \mathbf{pa}_{t_i}} Q(Ch_s \neq t_i|y)\{t_i = true\}\delta +$$

$$\left(1 - \prod_{Ch_s \in \mathbf{pa}_{t_i}} Q(Ch_s \neq t_i|y)\right) \{t_i = false\}\delta$$

Let

$$\begin{aligned}
&S(i, \mathbf{t}, y) = \\
&\quad \sum_{k \in bb(t_i,y)} Q(Ch_k = null|y) \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y) + \\
&\quad \sum_{k \in bb(t_i,y)} Q(Ch_k \neq null|y)(1 - \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y)) + \\
&\quad \sum_{k, body(k)[y]=true, \overline{t_i} \in body(k)} Q(Ch_k \neq null|y) + \\
&\quad \sum_{k, body(k)[y]=false, t_i \in body(k)} Q(Ch_k \neq null|y)
\end{aligned}$$

then

$$\begin{aligned}
&\mathbb{EP}'(t_i, y) = \\
&\quad \sum_{k \in bb(t_i,y)} \sum_{ch_k \neq null} Q(ch_k|y) \prod_{t_j \in body(k), t_j \neq t_i} Q(t_j|y) \log \theta_{Hd_{r(k)}=ch_k|true} + \\
&\quad \delta(S(i, t, y) + \prod_{Ch_s \in \mathbf{pa}_{t_i}} Q(Ch_s \neq t_i|y)\{t_i = true\}\delta + \\
&\quad (1 - \prod_{Ch_s \in \mathbf{pa}_{t_i}} Q(Ch_s \neq t_i|y))\{t_i = false\}\delta
\end{aligned}$$

So

$$\begin{aligned}
&\frac{\partial G_{t_i,y}(Q, \gamma)}{\partial \gamma} = \\
&\quad -\log Q(t_i) + \mathbb{EP}'(t_i, y) - \mathbb{E}_{Q(t'_i|y)}[\mathbb{EP}'(t'_i, y) - \log Q(t'_i)]
\end{aligned}$$

$$\mathbb{E}_{Q(ch_i|y)}[\log Q(ch_i)] = \sum_{ch_i} Q(ch_i|y) \log Q(ch_i)$$

$$\mathbb{E}_{Q(t_i|y)}[\log Q(t_i)] = \sum_{t_i} Q(t_i|y) \log Q(t_i)$$

□

Now that we have the optimization direction, we have to decide the size of the step to take.

Theorem 3 (Expression of $\mathbf{I}_Q(\mathbf{CH}, \mathbf{T}; Y)$).

$$\begin{aligned}
&\mathbf{I}_Q(\mathbf{CH}, \mathbf{T}; Y) = \\
&\quad \sum_i \sum_{ch_i} \sum_y Q(y)Q(ch_i|y)(\log Q(ch_i|y) - \log Q(ch_i)) + \\
&\quad \sum_i \sum_{t_i} \sum_y Q(y)Q(t_i|y)(\log Q(t_i|y) - \log Q(t_i))
\end{aligned}$$

Proof.

$$\begin{aligned}
&\mathbf{I}_Q(\mathbf{CH}, \mathbf{T}; Y) = \\
&\quad -\sum_i \mathbb{E}_Q[\log Q(CH_i)] - \sum_i \mathbb{E}_Q[\log Q(T_i)] + \sum_i \mathbb{E}_Q[\log Q(CH_i|Y)] + \sum_i \mathbb{E}_Q[\log Q(T_i|Y)] = \\
&\quad -\sum_i \sum_{ch_i, y} Q(ch_i|y)Q(y) \log Q(ch_i) - \sum_i \sum_{t_i, y} Q(t_i|y)Q(y) \log Q(t_i) + \\
&\quad \sum_i \sum_{ch_i, y} Q(ch_i|y)Q(y) \log Q(ch_i|y) + \sum_i \sum_{t_i, y} Q(t_i|y)Q(y) \log Q(t_i|y) = \\
&\quad \sum_i \sum_{ch_i} \sum_y Q(y)Q(ch_i|y) \log Q(ch_i|y) - Q(ch_i|y)Q(y) \log Q(ch_i) + \\
&\quad \sum_i \sum_{t_i} \sum_y Q(y)Q(t_i|y) \log Q(t_i|y) - Q(t_i|y)Q(y) \log Q(t_i) = \\
&\quad \sum_i \sum_{ch_i} \sum_y Q(y)Q(ch_i|y)(\log Q(ch_i|y) - \log Q(ch_i)) + \\
&\quad \sum_i \sum_{t_i} \sum_y Q(y)Q(t_i|y)(\log Q(t_i|y) - \log Q(t_i))
\end{aligned}$$

□

Theorem 4 (Derivatives of $\mathbf{I}_Q(\mathbf{CH}, \mathbf{T}; Y)$ with respect to $Q(ch_{i0}|y_0)$ and $Q(t_{i0}|y_0)$).

$$\frac{\partial \mathbf{I}_Q(\mathbf{CH}, \mathbf{T}; Y)}{\partial Q(ch_{i0}|y_0)} = Q(y_0)(\log Q(ch_{i0}|y_0) - \log Q(ch_{i0}))$$

$$\frac{\partial \mathbf{I}_Q(\mathbf{CH}, \mathbf{T}; Y)}{\partial Q(t_{i0}|y_0)} = Q(y_0)(\log Q(t_{i0}|y_0) - \log Q(t_{i0}))$$

Proof.

$$\frac{\partial \mathbf{I}_Q(\mathbf{CH}, \mathbf{T}; Y)}{\partial Q(ch_{i0}|y_0)} = -\frac{\partial \mathbb{E}_Q[\log Q(CH_i)]}{\partial Q(ch_{i0}|y_0)} + \frac{\partial \mathbb{E}_Q[\log Q(CH_i|y_0)]}{\partial Q(ch_{i0}|y_0)} =$$
$$-Q(y_0)(\log Q(ch_{i0}) + 1) + Q(y_0)(\log Q(ch_{i0}|y_0) + 1) =$$
$$Q(y_0)(-\log Q(ch_{i0}) - 1 + \log Q(ch_{i0}|y_0) + 1) =$$
$$Q(y_0)(\log Q(ch_{i0}|y_0) - \log Q(ch_{i0}))$$

$$\frac{\partial \mathbf{I}_Q(\mathbf{CH}, \mathbf{T}; Y)}{\partial Q(t_{i0}|y_0)} = -\frac{\partial \mathbb{E}_Q[\log Q(T_i)]}{\partial Q(t_{i0}|y_0)} + \frac{\partial \mathbb{E}_Q[\log Q(T_i|y_0)]}{\partial Q(t_{i0}|y_0)} =$$
$$-Q(y_0)(\log Q(t_{i0}) + 1) + Q(y_0)(\log Q(t_{i0}|y_0) + 1) =$$
$$Q(y_0)(-\log Q(t_{i0}) - 1 + \log Q(t_{i0}|y_0) + 1) =$$
$$Q(y_0)(\log Q(t_{i0}|y_0) - \log Q(t_{i0}))$$

□

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